Filtering time series

Randall Romero-Aguilar, February 28th, 2011

Based on:

- Cogley and Nason (1995) Effects of the Hodrick-Prescott filter on trend and difference stationary time series.
- Canova (1998) Detrending and business cycle facts.

The Real Business Cycle Model

- Kydland and Prescott use neoclassical growth model to study business cycle fluctuations:
- The steps in their methodology*:

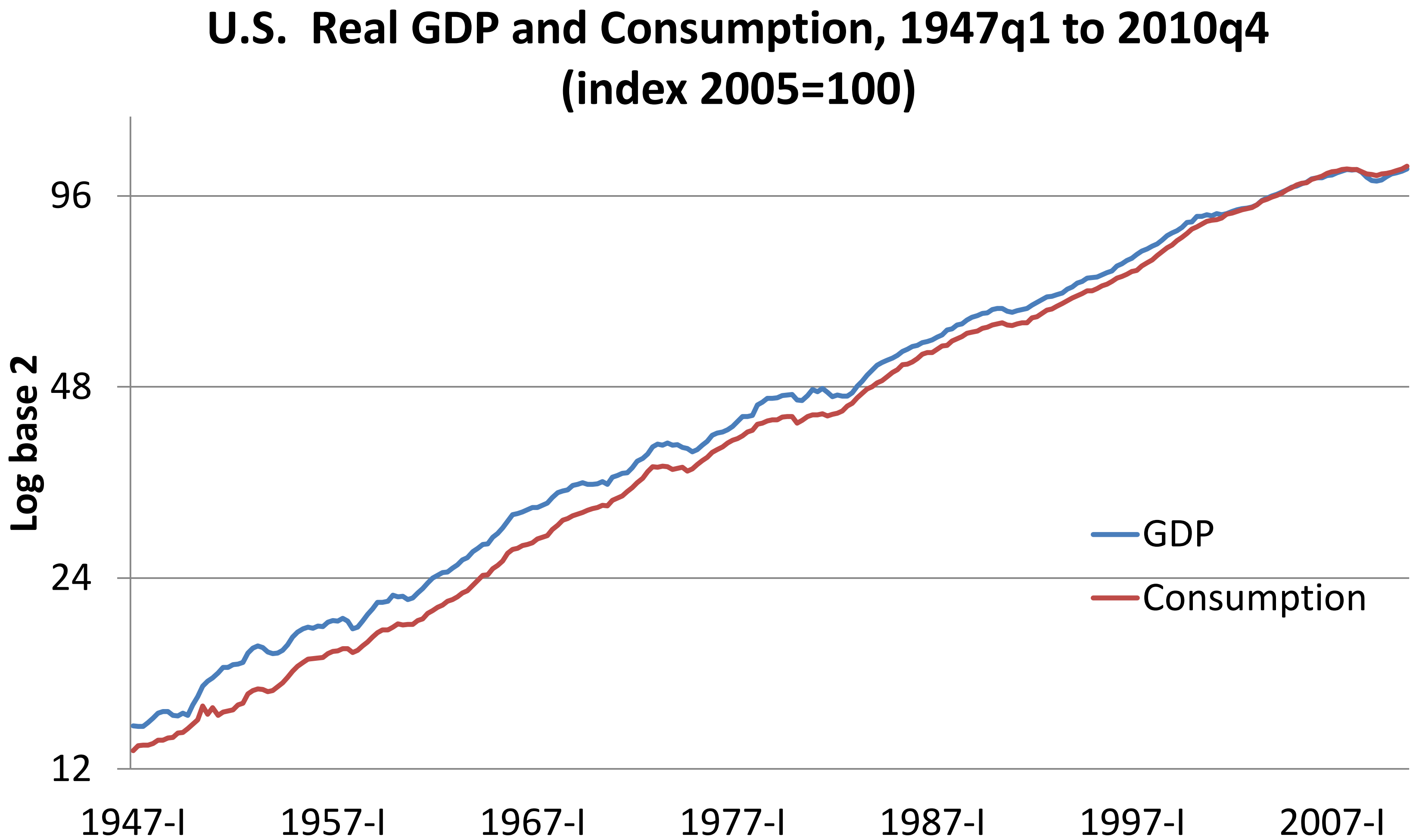
1. Start with the neoclassical growth model.
2. Modify the national accounts to be consistent with the theory.
3. Restrict the model to be consistent with the growth facts.
4. Introduce a Markovian shock process.
5. Make a linear-quadratic approximation.
6. Compute the competitive equilibrium process.
7. Simulate the model economy.
8. Examine the key business cycle statistics and draw scientific inferences.
9. Check for consistency with observations on individual households and firms.

* Prescott (2006). *Nobel Lecture: The Transformation of Macroeconomic Policy and Research*. The Journal of Political Economy, Vol. 114, No.2, pp.203-235

Canova (1998)

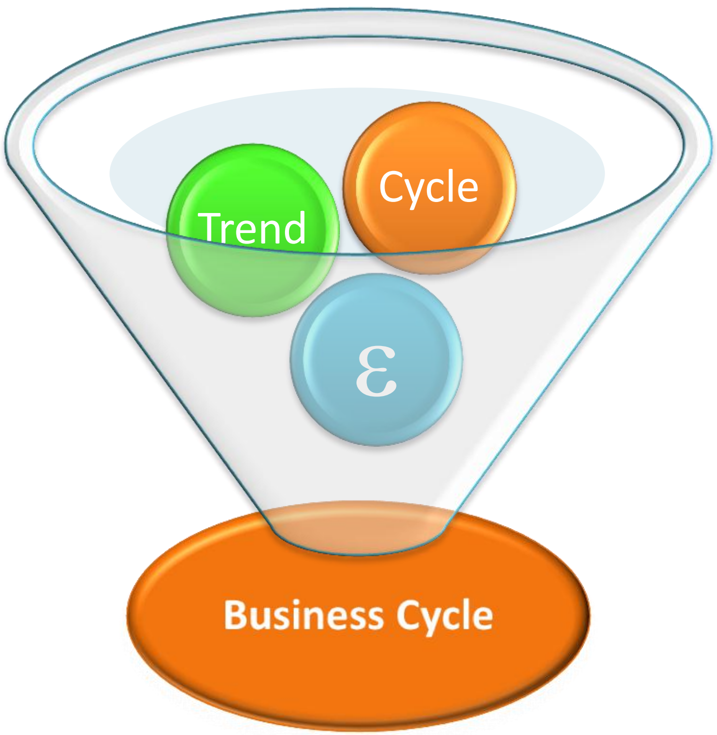
Detrending and business cycle facts.

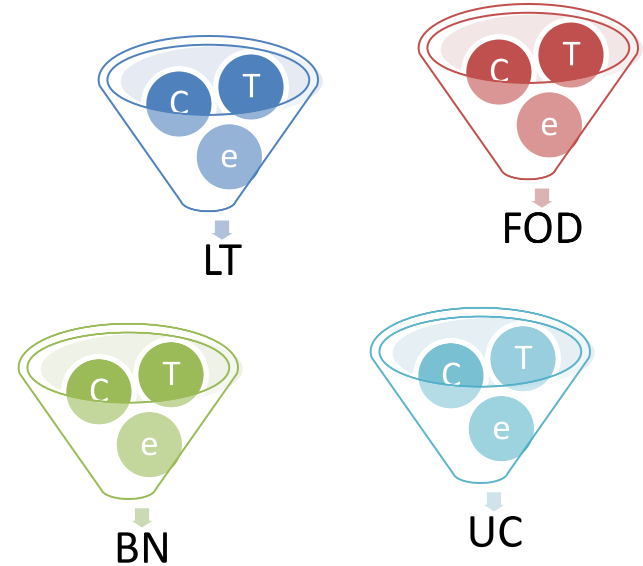
Problem: How to separate trend from cycle?

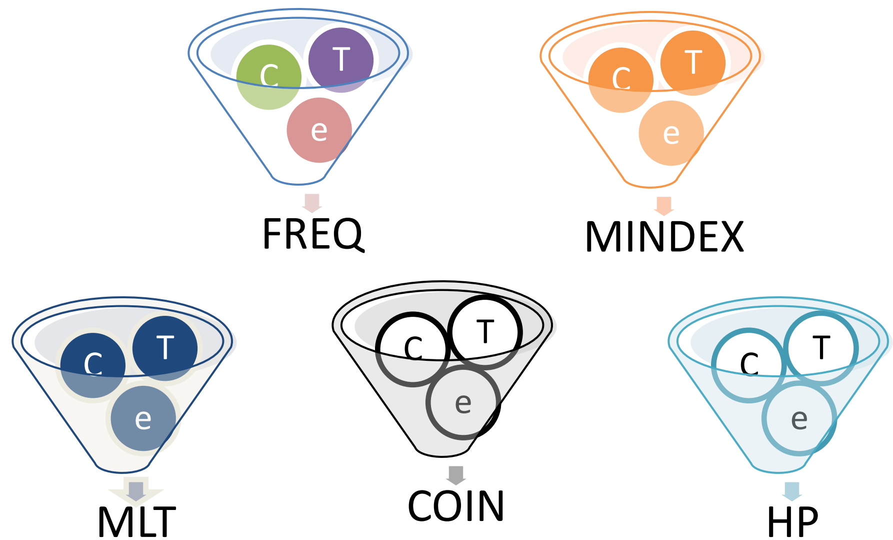


Source: Own computations, using data on BEA Table 1.1.3

Solution: Use a filter!







There are so many of them! Which filter should work better?

Canova (1998) main results

1. The practice of solely employing the HP1600 filter in compiling business cycle statistics is problematic.

2. The idea that there is a single set of facts which is more or less robust to the exact definition of business cycle is misleading.

3. The empirical characterization of the B.C. obtained with multivariate detrending methods is different from the one obtained with univariate procedures.

4. The practice of building theoretical models whose numerical versions quantitatively match one set of regularities obtained with a particular concept of cyclical fluctuation warrants a careful reconsideration.

Two problems connected with detrending

$$y_t = x_t + c_t$$

What are business cycles?

- Business cycles are deviations from the trend.
- But then, what is a trend?
- Are trends deterministic or stochastic processes?
- If stochastic, are they correlated to the cyclical component?

Statistic vs. Economic approach

- Concern on "measurement without theory".
- Dynamic economic theory does not indicate the type of economic trend that series may display nor the relationship between trend and cycle.
- Then, what is the "right" specification?

Alternative detrending methods

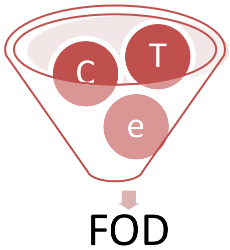
LT & SEGM	• Polynomial functions of time
FOD	• First order difference
BN	• Beveridge and Nelson's procedure
UC	• Unobserved components
FREQ1 & FREQ2	• Frequency domain methods
MINDEX	• One-dimension index model
MLT	• A model of common deterministic trends
COIN	• A model of common stochastic trends
HP 1600 & HP4	• The Hodrick and Prescott's filter

LT & SEGM

Polynomial function of time

- Trend and cycle uncorrelated.
- Trend is deterministic (polynomial of time).
- Trend estimated by OLS, residual gives cycle.

Linear trend	Linear trend with known structural break
$y_t = x_t + c_t$ $x_t = a + b\left(t - t_0\right)$	$y_t = x_t + c_t$ $x_t = \begin{cases} a_1 + b_1\left(t - t_0\right), & t \le \bar{t} \\ a_2 + b_2\left(t - \bar{t}\right), & t > \bar{t} \end{cases}$



First order differences

- The trend is a random walk with no drift.
- The cycle is stationary.
- Trend and cycle uncorrelated.
- Series has a unit root.

First order differences
$y_t = y_{t-1} + \varepsilon_t$ $x_t = y_{t-1}$ $c_t = \varepsilon_{t-1}$

Beveridge and Nelson's procedure

- Cycle is the stationary component of series.
- Trend is the non-stationary component.
- Trend and cycle perfectly correlated.
- Components derived from ARIMA process.

Beveridge and Nelson
$x_t = x_{t-1} + \mu + \left(\sum_{i=1}^{\infty} \gamma_i \right) \varepsilon_t$

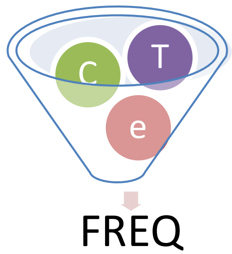
Unobserved components model

- Trend follows a random walk with drift.
- Cycle is a stationary finite order AR process.
- Trend and cycle might be correlated.

Unobserved components

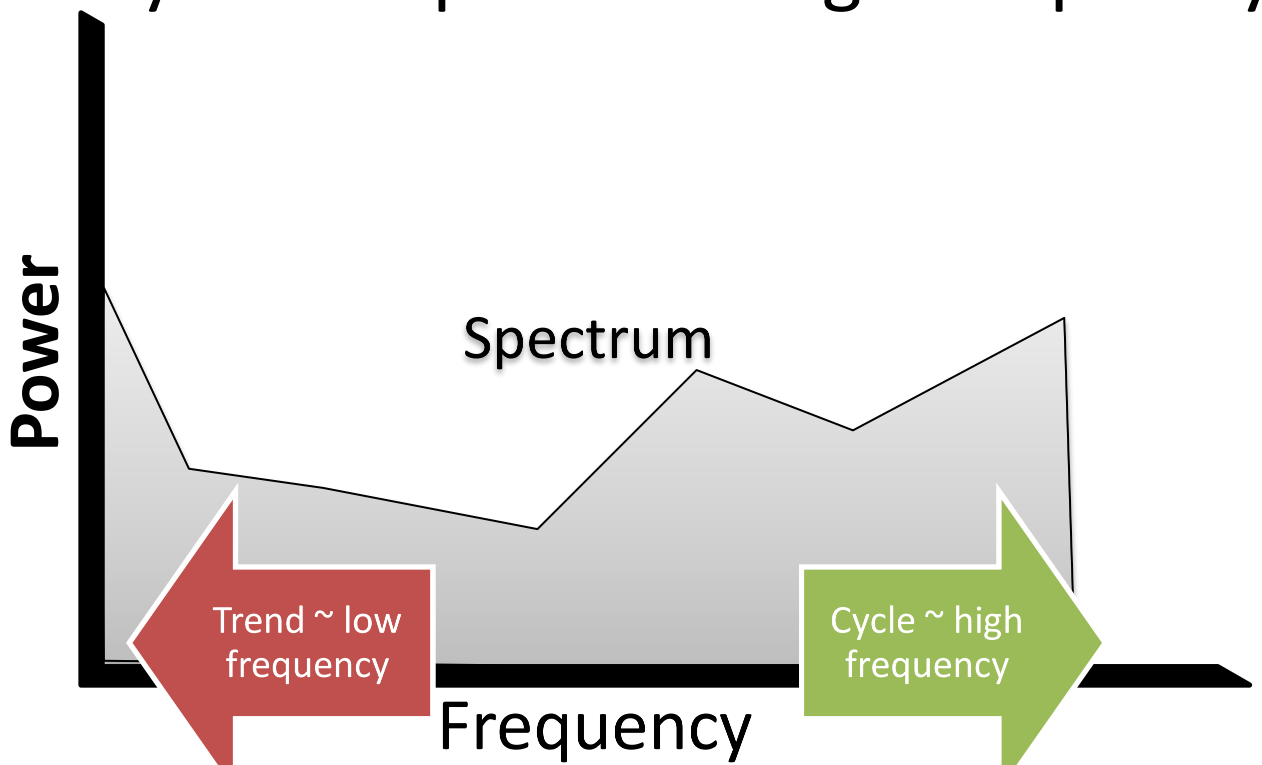
$$y_t = x_t + c_t + \varepsilon_t$$
$$x_t = x_{t-1} + \delta + u_t$$
$$c_t = \phi(L)c_{t-1} + v_t$$

$$[u_t, v_t] \sim N(0, \Sigma)$$
$$\varepsilon_t \sim N(0, \sigma^2), \quad E(\varepsilon_t \varepsilon_{t-j}) = 0$$
$$\varepsilon_t \text{ uncorrelated with } [u_t, v_t]$$



Frequency domain methods

- Trend and cycle are independent.
- Trend has power in low frequency of spectrum
- Cycle has power in high frequency of spectrum



MINDEX

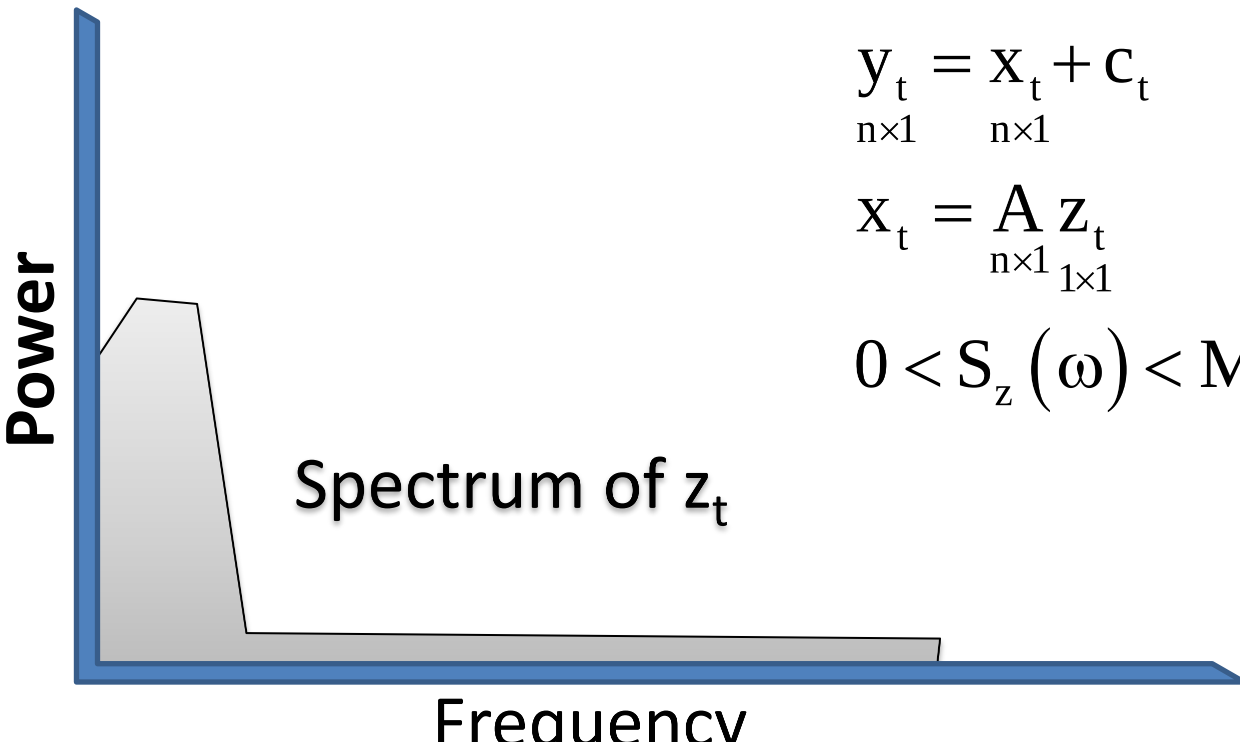
One-dimensional index model

- Multivariate method, while each series is trending, some linear combination of them does not have trend.

$$\underset{n\times 1}{y_t} = \underset{n\times 1}{x_t} + c_t$$

$$x_t = \underset{n\times 1}{A}\, \underset{1\times 1}{z_t}$$

$$0 < S_z(\omega) < M, \quad \forall \omega \in [\bar{\omega}, \pi]$$



Common deterministic trends

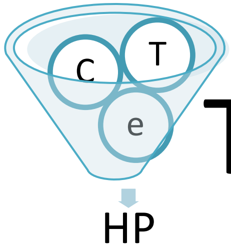
- Based on neoclassical model of capital accumulation, with deterministic labor augmenting technical change.
- All endogenous variables have a common deterministic trend.

Common deterministic trends
$\underset{n\times 1}{y_t} = \underset{n\times 1}{1}\,\underset{1\times 1}{x_t} + c_t$ $x_t = x_0 + \delta t$

Common stochastic trends

- Based on neoclassical model of capital accumulation, with non-stationary technological shock
- All endogenous variables have a common stochastic trend.

Common stochastic trends
$x_t = y_0 + A\tau_t = y_0 + \delta t + \psi\left(\varepsilon_1 + \ldots + \varepsilon_t\right)$

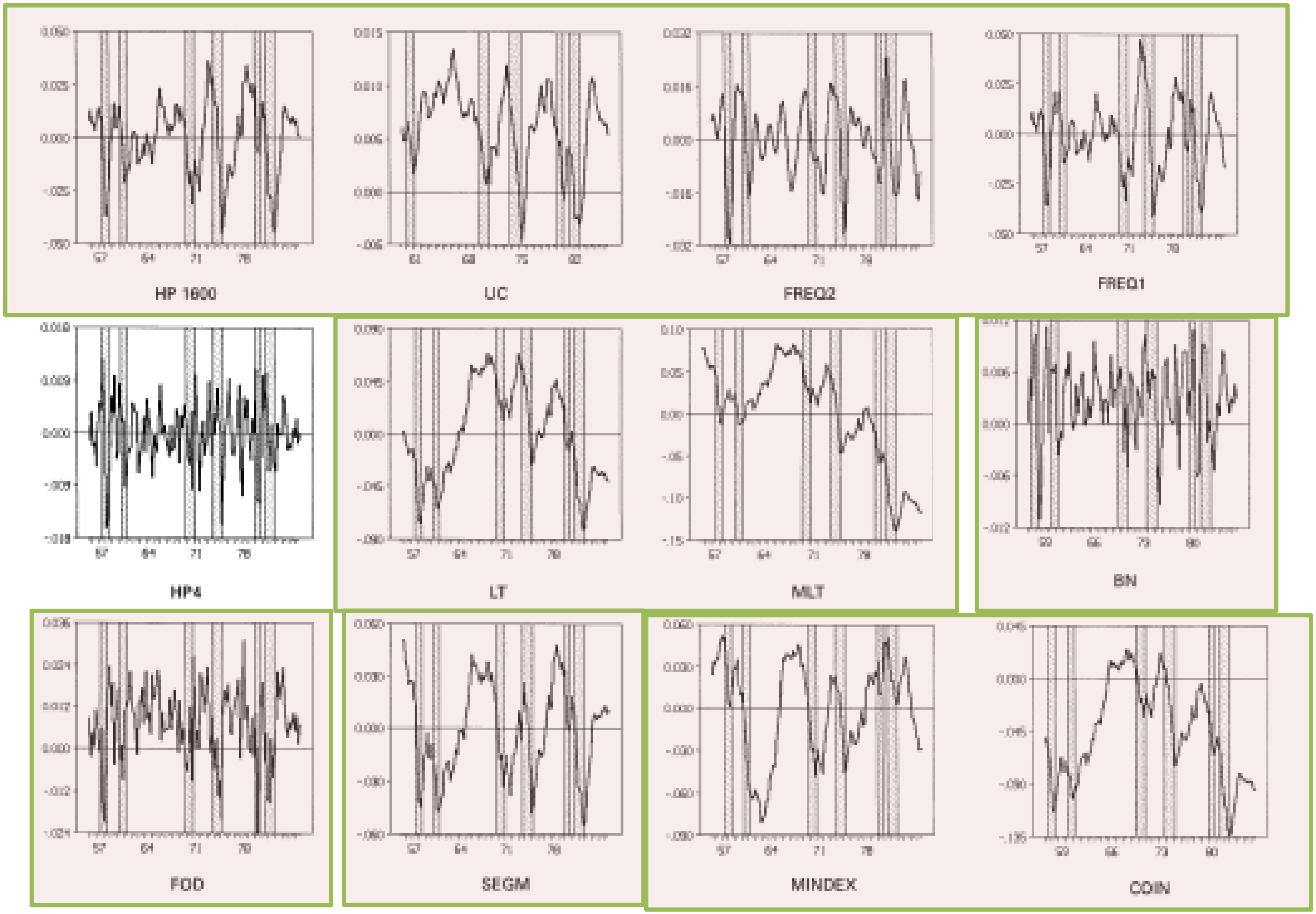


The Hodrick and Prescott's filter

- Trend is a representation of the preferences of the researcher and depends on question being investigated.
- Assumes that trend is smooth.

Hodrick and Prescott's filter
$\min_{x} \left\{ \sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=2}^{T-1} \left[\left(x_{t+1} - x_t\right) - \left(x_t - x_{t-1}\right) \right]^2 \right\}$

Cyclical components



Standard deviations

(absolute for GNP, all others relative to GNP)

Filter	GNP	Consumption	Investment	Hours	Real wage	Productivity	Capital
HP1600	1.76	0.49	2.82	1.06	0.70	0.49	0.61
HP4	0.55	0.48	2.70	0.89	0.65	0.69	0.14
FOD	1.03	0.51	2.82	0.91	0.98	0.67	0.63
BN	0.43	0.75	3.80	1.64	2.18	1.14	2.64
UC	0.38	0.34	6.72	4.14	2.24	4.09	1.22
LT	4.03	0.69	2.16	0.69	1.71	1.00	1.56
SEGM	2.65	0.52	3.09	1.01	1.10	0.54	0.97
FREQ1	1.78	0.46	3.10	1.20	1.07	0.66	1.41
FREQ2	1.14	0.44	3.00	1.16	1.11	0.69	1.26
MLT	6.01	0.67	2.36	0.46	1.21	1.00	1.05
MINDEX	3.47	0.98	2.65	1.14	1.27	0.72	1.85
COIN	4.15	0.71	3.96	0.75	1.68	1.09	1.30

Cross-correlations

(With respect to GNP, lags -1, 0 and 1)

	Consumption			Investment			Hours			Real wages		
	-1	0	1	-1	0	1	-1	0	1	-1	0	1
HP1600	0.75	0.75	0.62	0.76	0.91	0.84	0.67	0.88	0.90	0.81	0.81	0.63
HP4	0.16	0.31	0.01	0.07	0.65	0.28	0.11	0.73	0.44	0.12	0.49	-0.30
FOD	0.35	0.46	0.21	0.25	0.71	0.39	0.28	0.75	0.54	0.34	0.69	0.42
BN	0.23	0.42	0.38	-0.08	0.45	0.38	0.27	0.72	0.30	0.05	0.52	0.45
UC	0.79	0.74	0.61	0.82	0.45	0.73	0.01	0.17	0.28	0.80	0.85	0.79
LT	0.90	0.91	0.89	0.73	0.77	0.76	0.25	0.34	0.37	0.89	0.92	0.90
SEGM	0.76	0.81	0.76	0.78	0.86	0.79	0.71	0.85	0.86	0.79	0.88	0.84
FREQ1	0.75	0.73	0.57	0.73	0.86	0.80	0.63	0.83	0.85	0.73	0.84	0.75
FREQ2	0.68	0.69	0.52	0.58	0.85	0.85	0.48	0.80	0.87	0.59	0.81	0.79
MLT	0.93	0.96	0.93	-0.26	-0.26	-0.26	0.17	0.22	0.23	0.79	0.82	0.81
MINDEX	0.79	0.84	0.85	0.81	0.84	0.80	0.71	0.77	0.78	0.64	0.67	0.65
COIN	0.82	0.83	0.82	0.28	0.30	0.31	0.16	0.24	0.27	0.89	0.91	0.90

Impulse Response Function

(Summary statistics)

	Cycle length (quarters)	Size and location of peak response											
		Consumption		Investment		Hours		Real Wage		Productivity		Capital	
HP1600	20	2	0.28	3	1.93	3	0.76	1	0.46	1	2.00	6	0.30
HP4	8	1	0.17	1	1.50	1	0.58	1	0.37	1	1.70	2	0.05
FOD	6	1	0.25	1	1.50	1	0.57	1	0.53	1	1.82	4	0.11
BN	8	1	0.30	1	2.10	1	1.24	1	1.42	1	0.84	5	2.18
UC	21	1	0.23	1	6.02	5	2.38	4	1.18	1	0.98	6	0.54
LT	48	3	0.26	3	1.80	3	0.79	3	0.56	1	2.03	44	0.54
SEGM	19	1	0.24	4	1.86	3	0.77	1	0.52	1	2.02	6	0.36
FREQ1	17	4	0.30	3	1.98	3	0.82	3	0.60	1	2.10	4	0.31
FREQ2	12	4	1.12	4	10.20	4	3.25	4	2.08	4	2.06	25	1.53
MLT	48	2	0.26	2	1.84	3	0.79	3	0.52	1	1.83	7	0.28
MINDEX	39	2	0.28	4	1.91	2	0.81	3	0.55	1	1.67	44	0.36
COIN	24	3	1.32	4	6.23	4	3.18	6	1.78	1	0.74	10	0.42

Cogley and Nason (1995)

Effects of the Hodrick-Prescott filter on trend and difference stationary time series.

Some useful concepts

Stationary time series	$\{Y_t\}_{t=-\infty}^{\infty}$ such that $E(Y_t)=\mu$ and $V(Y_t)=\sigma^2, \quad \forall t$
Time domain representation	$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
Frequency domain representation	$Y_t = \mu + \int_0^{\pi} \left[\alpha(\omega)\cos(\omega t) + \delta(\omega)\sin(\omega t)\right] d\omega$
Autocovariance	$\gamma_j = \mathrm{cov}(Y_t, Y_{t-j}) = E\left[(Y_t - \mu)(Y_{t-j} - \mu)\right]$
Population spectrum	$s_Y(\omega) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j} = \frac{1}{2\pi}\left\{\gamma_0 + 2\sum_{j=1}^{\infty}\gamma_j \cos(\omega j)\right\}$
Its inverse:	$\gamma_k = \int_{-\pi}^{\pi} s_Y(\omega) e^{i\omega k} d\omega = \int_{-\pi}^{\pi} s_Y(\omega)\cos(\omega k)\, d\omega$

Some useful concepts (2)

Stationary time series	$\{S_t\}_{t=-\infty}^{\infty}, \quad E(S_t)=\mu \quad \wedge \quad V(S_t)=\sigma^2$
Trend-stationary (TS) time series	$\{X_t\}_{t=-\infty}^{\infty}, \quad X_t = \alpha + \beta t + S_t$
Difference-stationary (DS) time series	$\{X_t\}_{t=-\infty}^{\infty}, \quad X_t = X_{t-1} + S_t$
Lag polynomial	$(\alpha_p L^p + \ldots + \alpha_1 L + \alpha_0) X_t = \alpha_p X_{t-p} + \ldots + \alpha_1 X_{t-1} + \alpha_0 X_t$
Euler formula	$e^{i\omega} = \cos\omega + i\sin\omega$
Then:	$e^{i\omega} + e^{-i\omega} = 2\cos\omega \quad \wedge \quad e^{i\omega} - e^{-i\omega} = 2i\sin\omega$
Spectrum of a filter	$x_t = A(L) y_t \Rightarrow s_x(\omega) = A(e^{i\omega}) A(e^{-i\omega}) s_y(\omega)$

Spectral analysis: some examples

White noise

$$\left\{\varepsilon_t\right\}_{t=-\infty}^{\infty}$$

$$E\varepsilon_t = 0, \quad E\varepsilon_t^2 = \sigma^2$$

$$s_\varepsilon(\omega) = \frac{\sigma^2}{2\pi}$$

AR(1) process

$$y_t = \alpha + \rho y_{t-1} + \varepsilon_t$$

$$|\rho| < 1, \quad \varepsilon \sim WN\left(o, \sigma^2\right)$$

$$s_y(\omega) = \frac{\sigma^2}{2\pi\left(1+\rho^2 - 2\rho\cos\omega\right)}$$

More examples in Matlab

Cogley and Nason main result

When measuring the business component of a time series, is it a good idea to use the Hodrick-Prescott filter?

YES, if original series is Trend Stationary

NO, if original series is Difference-Stationary

The Hodrick-Prescott filter

$$\{g_t\}_{t=-\infty}^{\infty} = \arg\min \left\{ \sum_{j=-\infty}^{\infty} \left(y_{t+j} - g_{t+j}\right)^2 + \lambda \sum_{j=-\infty}^{\infty} \left(g_{t+j+1} - 2g_{t+j} + g_{t+j-1}\right)^2 \right\}$$

$$= \arg\min \left\{ \sum_{j=-\infty}^{\infty} \left(y_{t+j} - g_{t+j}\right)^2 + \lambda \sum_{j=-\infty}^{\infty} \left[\left(L^{-1} - 2 + L\right) g_{t+j}\right]^2 \right\}$$

F.O.C.

$$\lambda\left(L^{-1} - 2 + L\right)^2 g_t = y_t - g_t$$

$$\lambda\left(L^{-0.5} - L^{0.5}\right)^4 g_t = y_t - g_t$$

$$\left[1 + \lambda\left(L^{-0.5} - L^{0.5}\right)^4\right] g_t = y_t$$

$$g_t = \left[1 + \lambda\left(L^{-0.5} - L^{0.5}\right)^4\right]^{-1} y_t$$

$$c_t = \lambda\left(L^{-0.5} - L^{0.5}\right)^4 \left[1 + \lambda\left(L^{-0.5} - L^{0.5}\right)^4\right]^{-1} y_t$$

The Hodrick-Prescott filter (2)

Cyclical component of the HP filter

$$\boxed{c_t = \frac{\lambda\left(L^{-0.5} - L^{0.5}\right)^4}{1+\lambda\left(L^{-0.5} - L^{0.5}\right)^4} y_t = F_{HP}\left(L\right) y_t}$$

$$F_{HP}\left(L\right) = \frac{\lambda\left(L^{-0.5} - L^{0.5}\right)^4}{1+\lambda\left(L^{-0.5} - L^{0.5}\right)^4} = \frac{\lambda L^{-2}\left(1-L\right)^4}{1+\lambda\left(L^{-0.5} - L^{0.5}\right)^4} = \frac{\lambda A\left(L\right)}{1+\lambda A\left(L\right)}$$

Spectrum of the HP filter

Notice that: $A\left(L^{-1}\right)=\left(L^{0.5}-L^{-0.5}\right)^{4}=\left(L^{-0.5}-L^{0.5}\right)^{4}=A\left(L\right)$

$\rightarrow F_{HP}\left(L^{-1}\right)=F_{HP}\left(L\right)$

Furthermore:

$$A\left(e^{i\omega}\right)=\left(e^{-0.5\omega i}-e^{0.5\omega i}\right)^{4}=\left(-2i\sin\tfrac{\omega}{2}\right)^{4}=16\sin^{4}\left(\tfrac{\omega}{2}\right)$$

Then the spectrum of the filtered series is related to the spectrum of the original series by:

$$s_c\left(\omega\right)=F_{HP}\left(e^{i\omega}\right)F_{HP}\left(e^{-i\omega}\right)s_y\left(\omega\right)$$

$$=\left[F_{HP}\left(e^{i\omega}\right)\right]^{2}s_y\left(\omega\right)$$

$$\boxed{s_c\left(\omega\right)=\left[\frac{16\lambda\sin^{4}\left(\frac{\omega}{2}\right)}{1+16\lambda\sin^{4}\left(\frac{\omega}{2}\right)}\right]^{2}s_y\left(\omega\right)}$$

Non-stationary periodic decomposition?

- In practice, HP filter is applied to non-stationary time series (after all, it was designed to separate the trend from the cycle!)
- However, there is not a spectral representation for non-stationary processes!
- To study the effects of filtering trending series, think of the filter as a procedure that performs two operations:
  1. Render the series stationary by an appropriate transformation.
  2. Work on the stationary component.
- Then it is possible to analyze the effect of the filter on the cycle component by studying the second operation.
- Here we explored the effect of filtering on:
  1. Trend-stationary processes.
  2. Difference-stationary processes.

Filtering a TS process

$$y_t = \alpha + \beta t + z_t$$

$$F_{HP}(L) = \frac{\lambda L^{-2}(1-L)^4}{1+\lambda A(L)}$$

$$\begin{aligned}
z_t^{HP} &= F_{HP}(L) y_t \\
&= F_{HP}(L)(\alpha + \beta t) + F_{HP}(L) z_t \\
&= \frac{\lambda L^{-2}(1-L)^4}{1+\lambda A(L)}(\alpha + \beta t) + F_{HP}(L) z_t \\
&= \frac{\lambda L^{-2}(1-L)^2}{1+\lambda A(L)}(1-L)^2(\alpha + \beta t) + F_{HP}(L) z_t \\
&= F_{HP}(L) z_t
\end{aligned}$$

✓ Applying the HP filter to a TS process is equivalent to filtering only its stationary (cyclical) component.

Filtering a DS process

$$y_t = \alpha + y_{t-1} + z_t$$

$$z_t^{HP} = F_{HP}(L)\,y_t$$

$$= \frac{\lambda L^{-2}(1-L)^3}{1+\lambda A(L)}(1-L)^1 y_t$$

$$= \frac{\lambda L^{-2}(1-L)^3}{1+\lambda A(L)}(\alpha + z_t)$$

$$= \frac{\lambda L^{-0.5}(L^{-0.5}-L^{0.5})^3}{1+\lambda A(L)} z_t$$

$$= \frac{B(L)}{1+\lambda A(L)} z_t$$

$$B(L)B(L^{-1})$$

$$= \lambda L^{-0.5}(L^{-0.5}-L^{0.5})^3 \lambda L^{0.5}(L^{0.5}-L^{-0.5})^3$$

$$= -\left[\lambda(L^{0.5}-L^{-0.5})^3\right]^2$$

$$B(e^{i\omega})B(e^{-i\omega}) = -\left[\lambda(e^{0.5i\omega}-e^{-0.5i\omega})^3\right]^2$$

$$= -\left[\lambda\left(2i\sin\tfrac{\omega}{2}\right)^3\right]^2 = \left[8\lambda\sin^3\left(\tfrac{\omega}{2}\right)\right]^2$$

$$s_{z\_HP}(\omega) = \left[\frac{8\lambda\sin^3\left(\frac{\omega}{2}\right)}{1+16\lambda\sin^4\left(\frac{\omega}{2}\right)}\right]^2 s_z(\omega)$$

✓ Applying the HP filter to a DS process is NOT equivalent to filtering only its stationary (cyclical) component.

Square gain of the filter

DS process

Squared gain

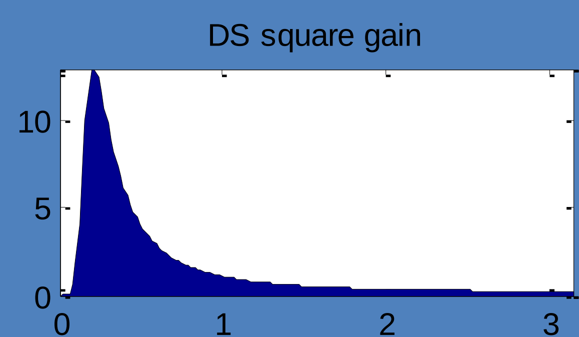
$$\left[\frac{8\lambda \sin^3\left(\frac{\omega}{2}\right)}{1+16\lambda \sin^4\left(\frac{\omega}{2}\right)}\right]^2$$

Maxima

max at $\omega = 2\sin^{-1}\left(\frac{1}{2}\sqrt[4]{3\lambda^{-1}}\right)$

equals $\dfrac{\sqrt{27\lambda}}{16}$

Plot



TS process

Squared gain

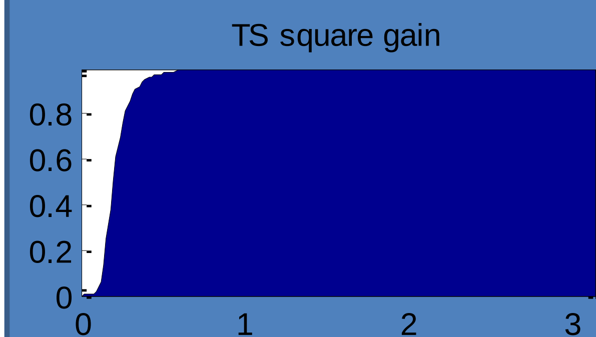
$$\left[\frac{16\lambda \sin^4\left(\frac{\omega}{2}\right)}{1+16\lambda \sin^4\left(\frac{\omega}{2}\right)}\right]^2$$

Maxima

maximum at $\omega = \pi$

equals $\left[\dfrac{16\lambda}{1+16\lambda}\right]^2$

Plot



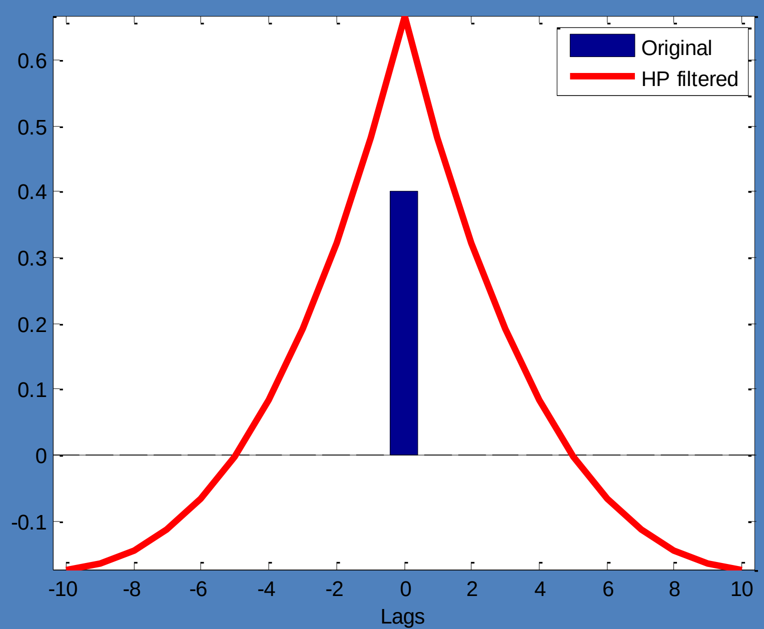
More examples in Matlab

Effects of HP filter on covariance

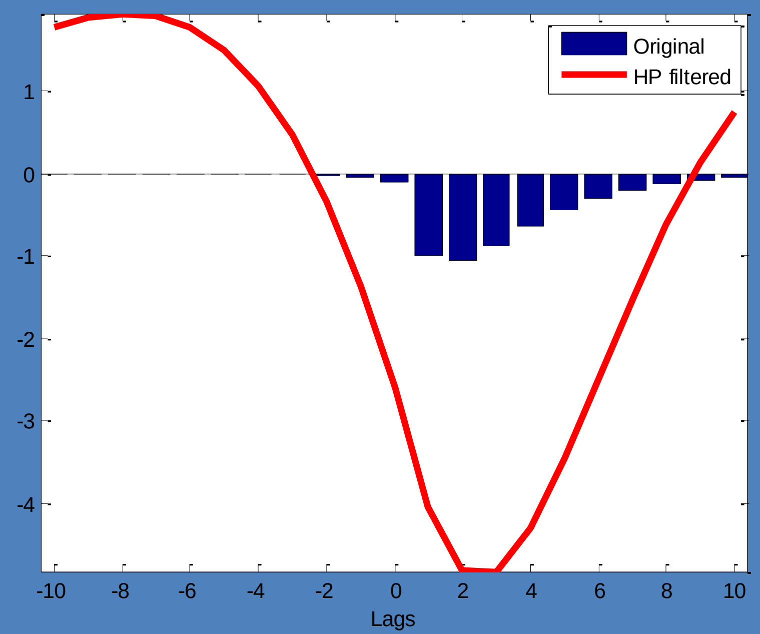
- Consider a VAR(1) process

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & .4 \\ .4 & 1 \end{bmatrix}$$



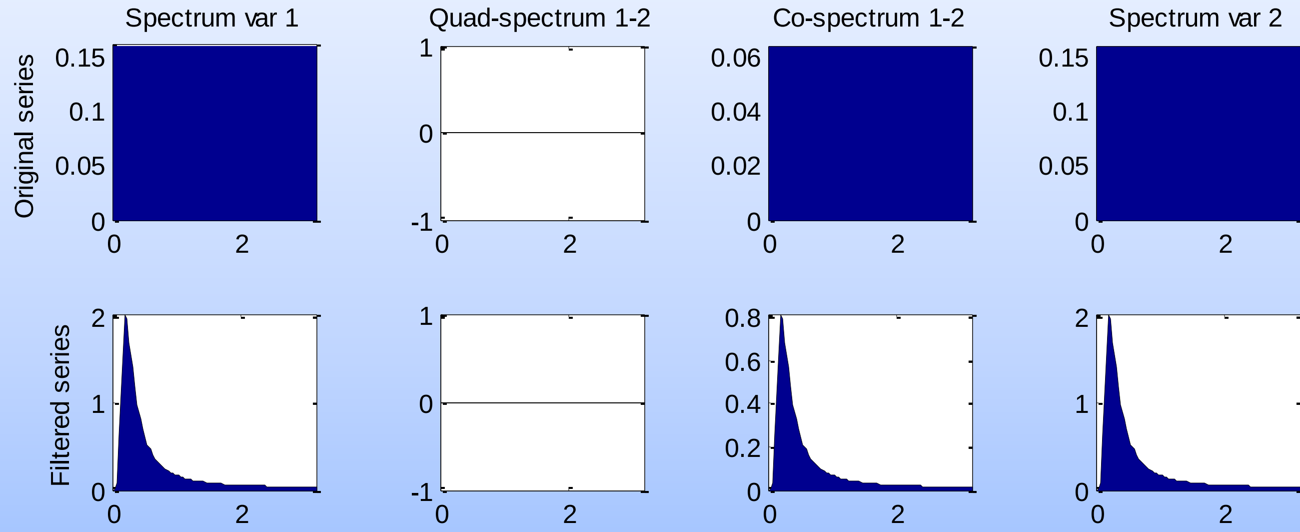
$$\Phi = \begin{bmatrix} .6 & -.7 \\ .0 & .5 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & .4 \\ .4 & 1 \end{bmatrix}$$



Effects of HP filter on spectra

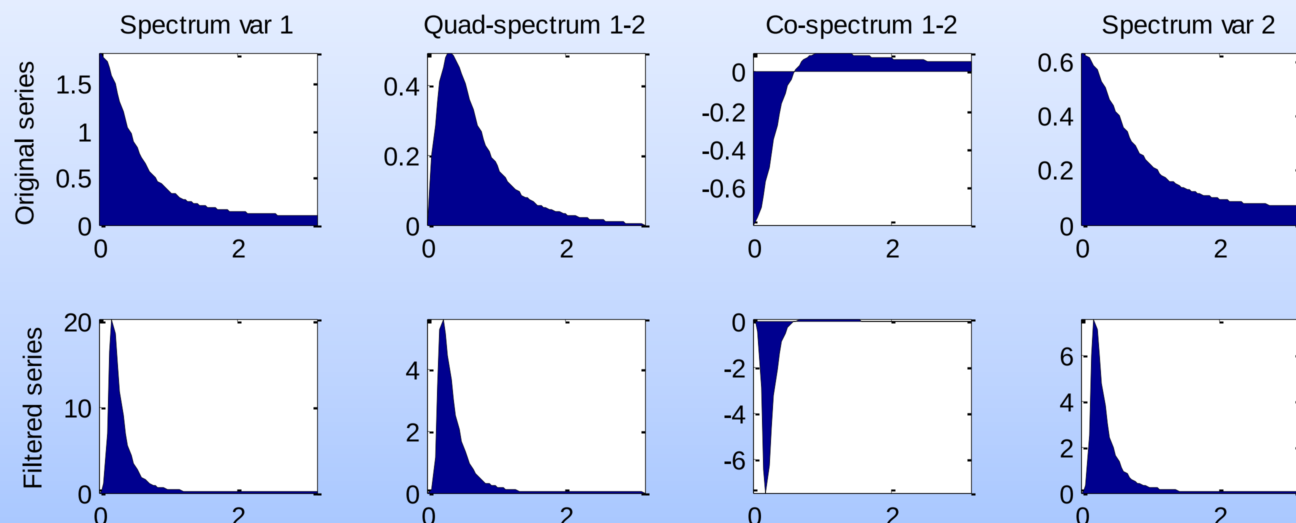
$$\Phi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & .4 \\ .4 & 1 \end{bmatrix}$$



$$\Phi = \begin{bmatrix} .6 & -.7 \\ .0 & .5 \end{bmatrix}$$

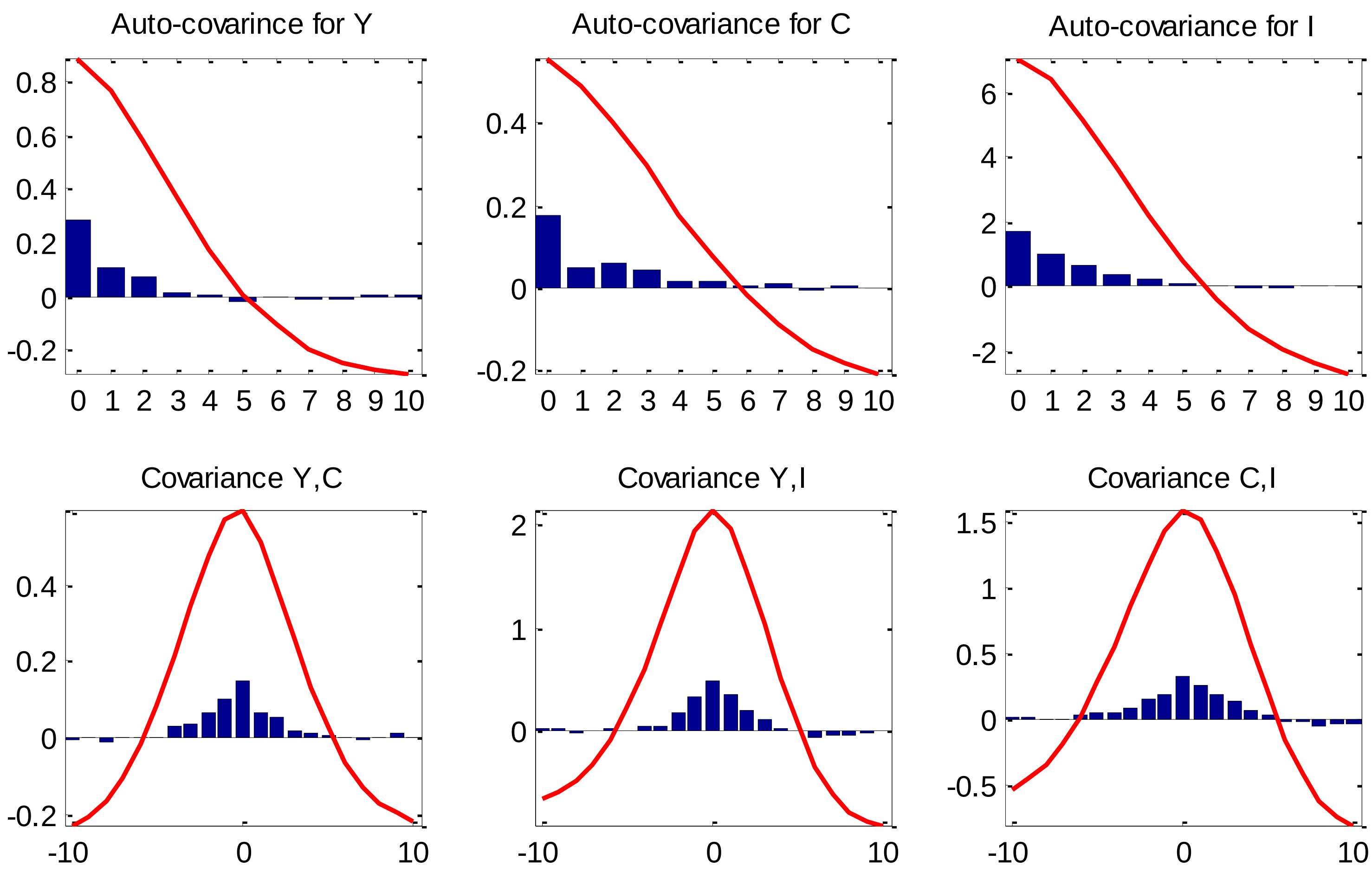
$$\Sigma = \begin{bmatrix} 1 & .4 \\ .4 & 1 \end{bmatrix}$$



Effects of HP on US data: covariance

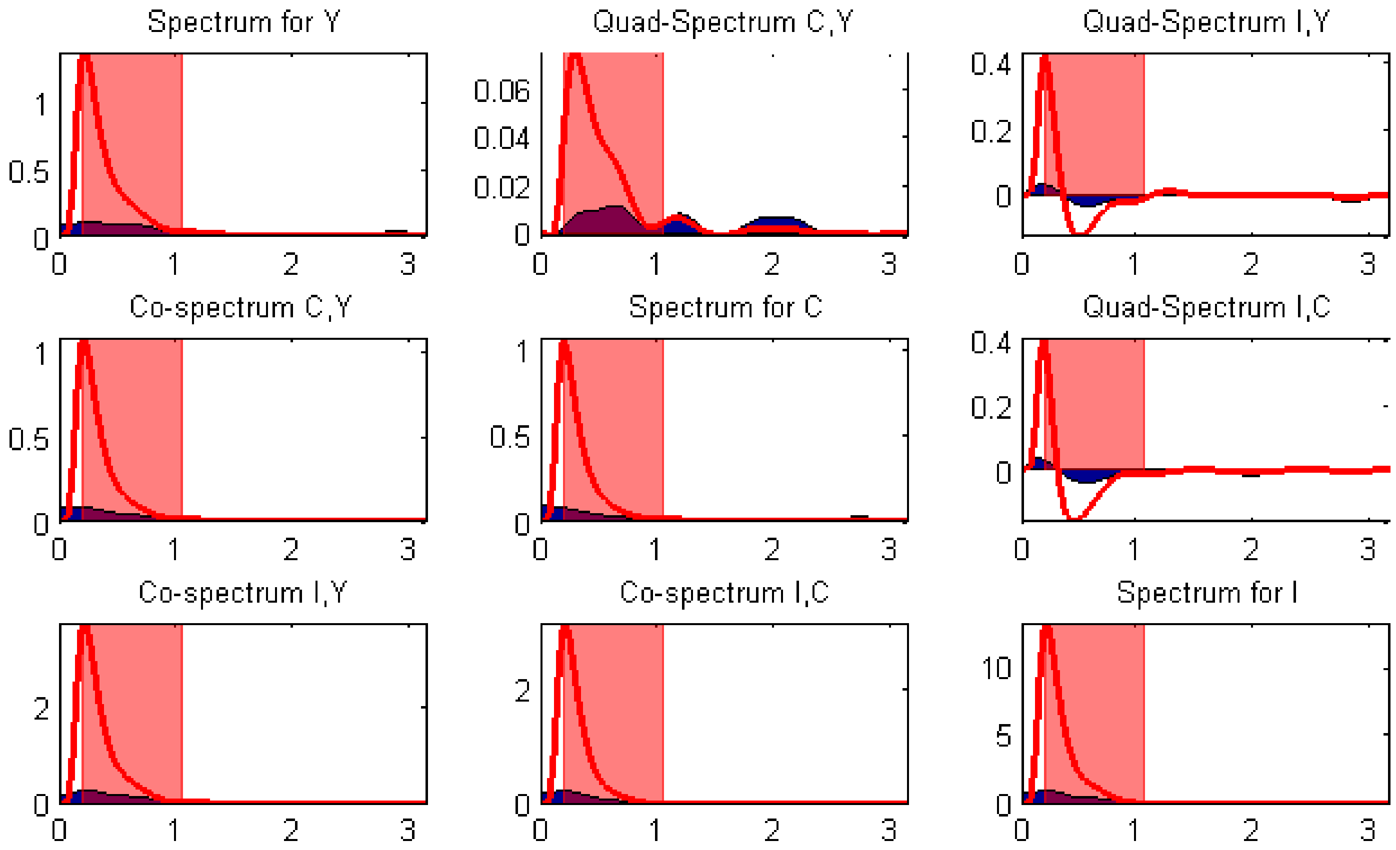
GDP, Consumption, and Investment, 1947-2010

- HP filtering greatly distorts the shape of the covariance function.



Effects of HP on US data: spectra

GDP, Consumption, and Investment, 1947-2010



HP filtered
Original

GDP growth: actual vs. simulated

(U.S. 1954-1991 vs Christiano-Eichenbaum (1992) model

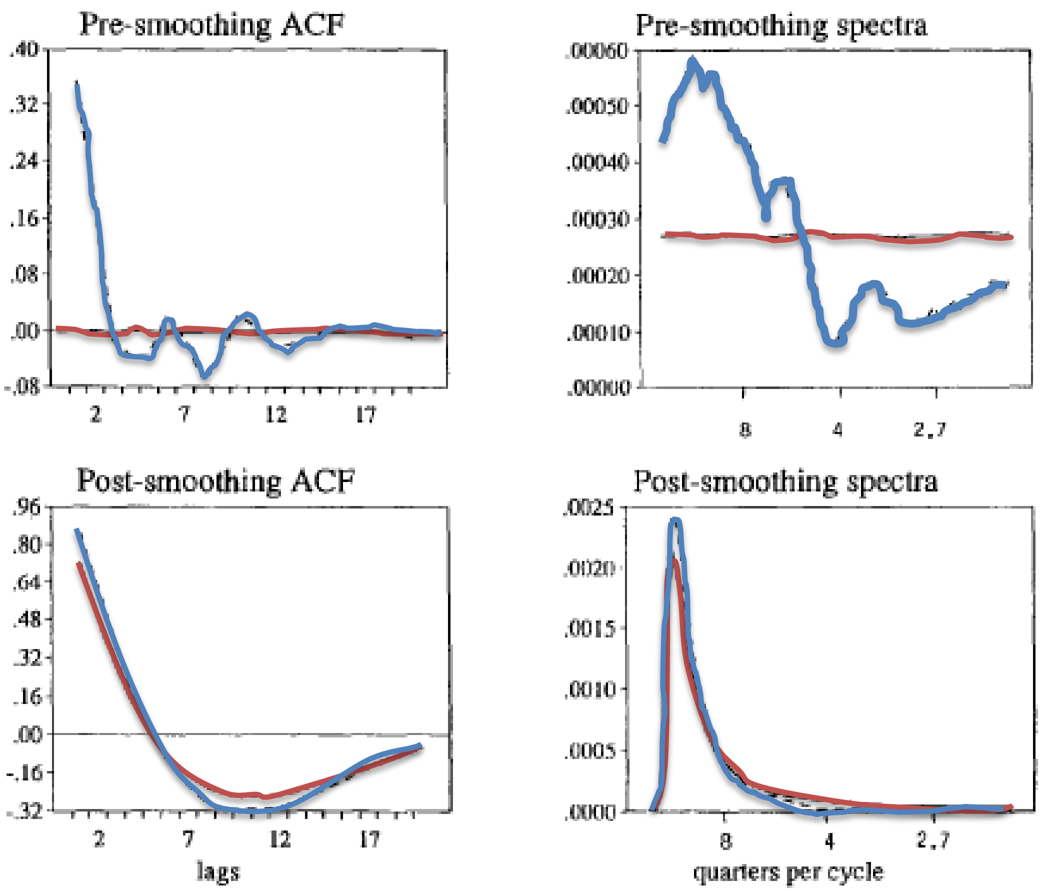
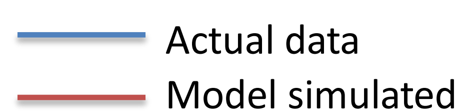


Fig. 4. Autocorrelations and power spectra, before and after filtering (DS model).

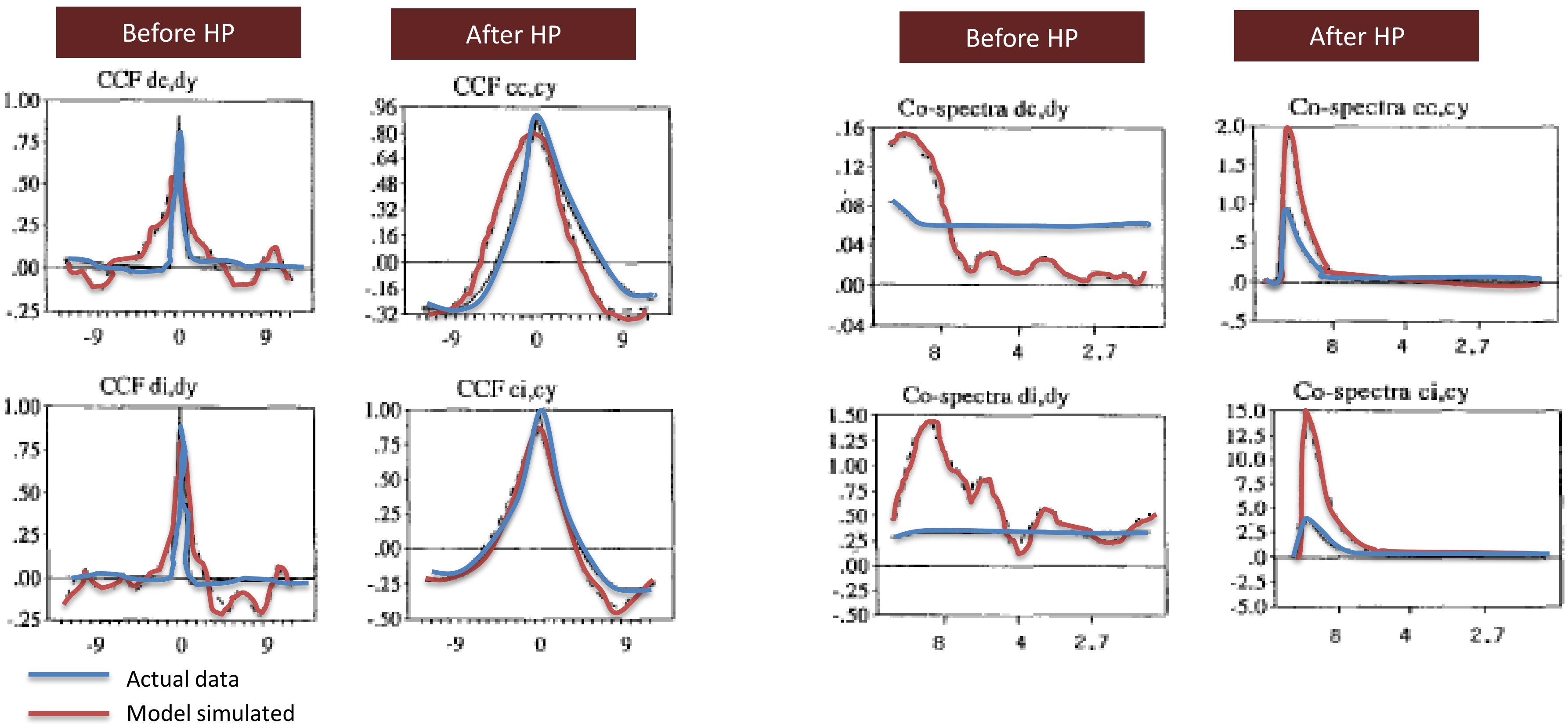
- Model does not generate business cycle periodicity.
- In this model, the source of business cycle periodicity in the HP filtered data is the HP filter itself!!



Comovement: actual vs. simulated

(U.S. 1954-1991 vs Christiano-Eichenbaum (1992) model

- An RBC model can exhibit business cycle comovements in HP filtered data even when pre-filter comovements are almost entirely contemporaneous.



Final remarks

If you apply the Hodrick-Prescott filter to a Difference-Stationary process, you may obtain completely biased results when evaluating a RBC model.

There are many filters to choose from when separating the trend from the cycle in a time series.

Unfortunately, “stylized facts” about the business cycle are not robust to the choice of filter.