A note on Chebyshev polynomials

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Chebyshev polynomials are very useful for interpolating functions. Formally, the Chebyshev polynomial of degree n is defined as

$$T_n(x) = \cos(n \cos^{-1} x), \qquad \text{for } x \in [-1,\ 1]$$

At first look, this expression does not resemble a polynomial at all!

In this note we will follow two different approaches to show that $T_n(x)$ is indeed a polynomial. We start with the easy one, which only requires some basic trigonometric identities in *real* numbers, and where we will find a *recursive* definition of the Chebyshev polynomials. The not-so-easy approach requires working with *complex* numbers, but it will give us a *closed* formula for computing the polynomials.

The easy path

Using the definition of the Chebyshev polynomials and a couple of trigonometric results we can find a convenient recursion:

$$\begin{aligned} T_{n+1}(x) + T_{n-1}(x) &= \cos[(n+1)\cos^{-1} x] + \cos[(n-1)\cos^{-1} x] \\ &= \cos[n\cos^{-1} x + \cos^{-1} x] + \cos[n\cos^{-1} x - \cos^{-1} x] \end{aligned}$$

apply the identity $\cos(u+v) + \cos(u-v) = 2\cos u \cos v$ to get

$$\begin{aligned} &= 2\cos(n\cos^{-1} x)\cos(\cos^{-1} x) \\ &= 2xT_n(x) \end{aligned}$$

That is, given two consecutive Chebyshev polynomials, the next one can be defined by the recursion

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

The first two elements of the sequence can be computed easily:

$$\begin{aligned} T_0(x) &= \cos(0\cos^{-1} x) = 1 \\ T_1(x) &= \cos(1\cos^{-1} x) = x \end{aligned}$$

Therefore, an alternative definition, where the polynomials are more evident, is given by the recursion:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), \qquad \text{for } n = 1, 2, \ldots \end{aligned}$$

The challenging path

In this path, we will need the binomial and de Moivre's formulas.

The binomial formula for complex numbers is a straightforward generalization of the corresponding formula for real numbers:

Theorem 1 (Binomial formula for complex numbers) *For z_1 and z_2 complex numbers*

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k}\, z_2^k$$

De Moivre's formula is easily derived from Euler's formula $e^{ix} = \cos(nx) + i\sin(nx)$, where i is the imaginary unit.

Theorem 2 (de Moivre's formula)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + \sin n\theta$$

We are now ready to show that $T_n(x)$ is a polynomial.

Theorem 3 (Chebyshev polynomials) *For $-1 \le x \le 1$ the expression*

$$T_n(x) = \cos(n\cos^{-1} x)$$

is a polynomial of degree n in the variable x.

Proof Let's define the integer m as:

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Start with de Moivre's and the binomial formulas

$$\begin{aligned}
\cos n\theta + i\sin n\theta &= (\cos\theta + i\sin\theta)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta\, (i\sin\theta)^k \\
&= \sum_{k=0,\text{even}}^{n} \binom{n}{k} \cos^{n-k}\theta\; i^k \sin^k\theta + \sum_{k=0,\text{odd}}^{n} \binom{n}{k} \cos^{n-k}\theta\; i^k \sin^k\theta \\
&= \sum_{k=0}^{m} \binom{n}{2k} \cos^{n-2k}\theta\; i^{2k} \sin^{2k}\theta + \sum_{k=0}^{m} \binom{n}{2k+1} \cos^{n-2k-1}\theta\; i^{2k+1} \sin^{2k+1}\theta \\
&= \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta\;\; \sin^{2k}\theta + \sum_{k=0}^{m} \binom{n}{2k+1} (-1)^k i \cos^{n-2k-1}\theta\;\; \sin^{2k+1}\theta
\end{aligned}$$

Equating the real parts:

$$\begin{aligned}\cos n\theta &= \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \ \sin^{2k}\theta \\ &= \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \ (1-\cos^2\theta)^k \\ &= \sum_{k=0}^{m} \binom{n}{2k} \cos^{n-2k}\theta \ (\cos^2\theta - 1)^k\end{aligned}$$

Write $x = \cos\theta$ and suppose that $0 \le \theta \le \pi$, in which case $-1 \le x \le 1$ and $\theta = \cos^{-1} x$. Substitute

$$\cos(n\cos^{-1} x) = \sum_{k=0}^{m} \binom{n}{2k} x^{n-2k} \ (x^2-1)^k$$

using the binomial formula once more

$$\begin{aligned}&= \sum_{k=0}^{m} \left[\binom{n}{2k} x^{n-2k} \ \sum_{h=0}^{k} \binom{k}{h} (-1)^h (x^2)^{k-h} \right] \\ &= \sum_{k=0}^{m} \sum_{h=0}^{k} \binom{n}{2k} \binom{k}{h} (-1)^h x^{n-2h}\end{aligned}$$

From this formula it is clear that the highest degree monomials are obtained when $h = 0$. To complete the proof, all we need to show is that the x^n monomials do not cancel each other: setting $h = 0$ we obtain the leading coefficient α_n:

$$\begin{aligned}\alpha_n &= \sum_{k=0}^{m} \binom{n}{2k} \binom{k}{0} (-1)^0 \\ &= \sum_{k=0}^{m} \binom{n}{2k} \\ &= 2^{n-1}\end{aligned}$$

Useful properties

Roots and extrema

Since $T_n(x) = \cos(n\cos^{-1} x)$, it is clear that $-1 \le T_n(x) \le 1$ (remember that the polynomial is only defined for $x \in [-1, 1]$). Extrema occurs when $n\cos^{-1} x = k\pi$ for some integer k:

$$x = \cos\left(\frac{k}{n}\pi\right), \qquad k = 0, \dots, n$$

On the other hand, the zeros of the polynomial occur when $n\cos^{-1} x = k\pi + \frac{\pi}{2}$ for some integer k:

$$x = \cos\left(\frac{2k-1}{2n}\pi\right), \qquad k = 1, \dots, n$$

Orthogonality

—TO BE COMPLETED—

References

Brown and Churchill (1996) *Complex Variables and Applications*, 6^{th} edition. McGraw-Hill.