Examples of the Central Limit Theorem

Randall Romero Aguilar

First draft: September 15, 2013
This draft: September 25, 2014

In this short note I provide proofs of the Central Limit Theorem (CLT). In plain words, the CLT states that the sample mean of a sufficiently large sample of independent random variables (from **any** distribution with finite mean and variance) will be approximately **normally** distributed.

In the first section I define the characteristic function (c.f.) of a random variable and present some of its properties. In the second section, I prove the CLT for the exponential (a continuous) and the Poisson (a discrete) distributions. The objective here is to emphasize the beauty of the CLT: it applies to *any* distribution with finite mean and variance. After illustrating the proof of the CLT for these two distributions, I show how the same procedure can be used to prove the general result.

1 The characteristic function

Definition Let X be a random variable. Its *characteristic function* $\phi_X(t)$ is a function of t given by

$$\phi_X(t) = \mathbb{E}\, e^{itX} \tag{1}$$

where $i = \sqrt{-1}$ is the imaginary unit.

The normal distribution appears in the CLT, so let's derive its c.f.

Example Let X be a normal random variable with mean μ and variance σ^2. A quick and dirty way to find its characteristic function is:

$$\begin{aligned}
X &\sim N\left(\mu, \sigma^2\right) \\
itX &\sim N\left(it\mu, i^2t^2\sigma^2\right) \\
e^{itX} &\sim logN\left(it\mu, -t^2\sigma^2\right) && \text{(definition of lognormal, } i^2 = -1) \\
\phi_X(t) = \mathbb{E}\, e^{itX} &= \exp\left\{it\mu - 0.5t^2\sigma^2\right\} && \text{(expected value of lognormal)}
\end{aligned}$$

More formally

$$
\begin{aligned}
\mathbb{E}\, e^{itX} &= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{it\mu - \frac{1}{2}t^2\sigma^2} \frac{1}{\sigma\sqrt{2\pi}} e^{itx - it\mu + \frac{1}{2}t^2\sigma^2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, \mathrm{d}x \\
&= e^{it\mu - \frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu-it\sigma^2)^2}\, \mathrm{d}x \\
&= e^{it\mu - \frac{1}{2}t^2\sigma^2}
\end{aligned}
$$

where the last integral equals one because is the sum of the pdf for a $N(\mu + it\sigma^2, \sigma^2)$ distribution. In particular, for the standard normal distribution $Z \sim N(0,1)$ it follows that $\phi_Z(t) = \exp\left\{-0.5t^2\right\}$. ∎

Some useful properties of the characteristic function are:

Theorem 1 *Two independent random variables X and Y have the same distribution if and only if they have the same characteristic function.*

Proposition 2 *Let X, Y be two independent random variables and a, b two constants. Then*

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) \tag{2}$$

$$\phi_a(t) = e^{iat} \tag{3}$$

$$\phi_{aX}(t) = \phi_X(at) \tag{4}$$

Proof The last two results simply follow from the definition of c.f. In the case of (3):

$$\phi_a(t) = \mathbb{E}\, e^{iat} = e^{iat} \quad (\text{since } a \text{ is a constat})$$

and in the case of (4):

$$\phi_{aX}(t) = \mathbb{E}\, e^{i(aX)t} = \mathbb{E}\, e^{iX(at)} = \phi_X(at)$$

For (2):

$$
\begin{aligned}
\phi_{X+Y}(t) &= \mathbb{E}\, e^{i(X+Y)t} \\
&= \mathbb{E}\left(e^{iXt} e^{iYt}\right) \\
&= \mathbb{E}\, e^{iXt}\, \mathbb{E}\, e^{iYt} \qquad X, Y \text{ are independent} \\
&= \phi_X(t)\phi_Y(t)
\end{aligned}
$$

The following lemma can be proved by using the results (2)-(4)

Lemma 3 *Let $X_1, X_2, \ldots, X_n$ be n identical and independent random variables, with common finite moments $\mathbb{E}\, X = \mu$ and $\mathbb{E}\left(X - \mu\right)^2 = \sigma^2$ and characteristic function $\phi_X(t)$. Let Y be defined by*

$$Y = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} \left(X_i - \mu\right)$$

Then:

$$\phi_Y(t) = \left[\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n \exp\left\{\frac{-i\mu t\sqrt{n}}{\sigma}\right\} \tag{5}$$

2 Proving the Central Limit Theorem

The examples below follow the definitions and conditions in lemma 3.

2.1 The exponential distribution

For the exponential random variable with cdf $F(x) = 1 - e^{\frac{-x}{\beta}}$, the expected value and the standard deviation are given by $\mu = \sigma = \beta$, and the c.f. is $\phi_X(t) = \left(1 - \beta t\right)^{-1}$. Using lemma 3

$$\phi_Y(t) = \left[1 - \beta\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^{-n} \exp\left\{\frac{-i\mu t\sqrt{n}}{\sigma}\right\}$$

but since $\mu = \sigma = \beta$

$$= \left[1 - \frac{t}{\sqrt{n}}\right]^{-n} \exp\left\{-it\sqrt{n}\right\}$$

using the transformation $a^b = \exp[b\log(a)]$

$$\begin{aligned} &= \exp\left\{-n\log\left(1 - \frac{t}{\sqrt{n}}\right)\right\}\exp\left\{-it\sqrt{n}\right\} \\ &= \exp\left\{-n\log\left(1 - \frac{t}{\sqrt{n}}\right) - it\sqrt{n}\right\} \end{aligned}$$

Taking the limits on both sides as $n \to \infty$ and defining $m = \frac{t}{\sqrt{n}} \Rightarrow n = \frac{t^2}{m^2}$

$$\begin{aligned} \lim_{n\to\infty}\phi_Y(t) &= \lim_{n\to\infty}\exp\left\{-n\log\left(1 - \frac{t}{\sqrt{n}}\right) - it\sqrt{n}\right\} \\ &= \lim_{m\to 0}\exp\left\{-\frac{t^2}{m^2}\log\left(1-m\right) - \frac{it^2}{m}\right\} \\ &= \exp\left\{-t^2\lim_{m\to 0}\frac{\log\left(1-m\right) + im}{m^2}\right\} \end{aligned}$$

this limit has the indeterminate form $\frac{0}{0}$, so we can apply L'Hôpital's Rule:

$$\begin{aligned} &= \exp\left\{-t^2\lim_{m\to 0}\frac{1}{2(1-m)}\right\} \\ &= \exp\left\{-\frac{t^2}{2}\right\} \end{aligned}$$

which is the characteristic function of the standard normal distribution.

2.2 The Poisson distribution

For the Poisson random variable with pmf $f(X = x) = \frac{e^{-x}\lambda^x}{x!}$, the expected value and the variance are given by $\mu = \sigma^2 = \lambda$, and the c.f. is $\phi_X(t) = \exp\left\{\lambda\left(e^{it} - 1\right)\right\} = \exp\left\{\lambda e^{it} - \lambda\right\}$. Using lemma 3

$$
\begin{aligned}
\phi_Y(t) &= \exp\left\{\lambda \exp\left(i\frac{t}{\sigma\sqrt{n}}\right) - \lambda\right\}^n \exp\left\{\frac{-i\mu t\sqrt{n}}{\sigma}\right\} \\
&= \exp\left\{\lambda n \exp\left(i\frac{t}{\sigma\sqrt{n}}\right) - \lambda n - \frac{i\mu t\sqrt{n}}{\sigma}\right\}
\end{aligned}
$$

but since $\mu = \sigma^2 = \lambda$

$$
= \exp\left\{\lambda n \exp\left(\frac{it}{\sqrt{\lambda n}}\right) - \lambda n - it\sqrt{\lambda n}\right\}
$$

Define $m = \frac{t}{\sqrt{\lambda n}} \Rightarrow \lambda n = \frac{t^2}{m^2}$, substitute

$$
= \exp\left\{\frac{t^2\exp(im) - t^2 - imt^2}{m^2}\right\}
$$

Taking the limits on both sides as $n \to \infty$

$$
\begin{aligned}
\lim_{n\to\infty}\phi_Y(t) &= \lim_{m\to 0}\exp\left\{\frac{t^2\exp(im) - t^2 - imt^2}{m^2}\right\} && \text{(apply L'Hôpital)} \\
&= \exp\left\{\frac{it^2}{2}\lim_{m\to 0}\frac{\exp(im) - 1}{m}\right\} && \text{(once more)} \\
&= \exp\left\{\frac{it^2}{2}\lim_{m\to 0} i\exp(im)\right\} = \exp\left\{-\frac{t^2}{2}\right\}
\end{aligned}
$$

which is the characteristic function of the standard normal distribution.

2.3 General proof

We need to show that

$$\lim_{n\to\infty} \phi_Y(t) = \lim_{n\to\infty} \left[\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n \exp\left\{\frac{-i\mu t\sqrt{n}}{\sigma}\right\} = \exp\left\{-\tfrac{1}{2}t^2\right\} \tag{6}$$

Let's start with the change of variable $m = \frac{t}{\sigma\sqrt{n}} \Rightarrow n = \frac{t^2}{m^2\sigma^2}$. To evaluate the limit as $n \to \infty$ is equivalent to evaluating as $m \to 0$

$$\begin{aligned}
\lim_{n\to\infty} \phi_Y(t) &= \lim_{m\to 0} \phi_Y(t) \\
&= \lim_{m\to 0} \left[\phi_X(m)\right]^{\frac{t^2}{\sigma^2 m^2}} \exp\left\{\frac{-i\mu t^2}{\sigma^2 m}\right\}
\end{aligned}$$

using the transformation $a^b = \exp[b\log(a)]$

$$\begin{aligned}
&= \lim_{m\to 0} \exp\left\{\frac{t^2}{\sigma^2 m^2}\log\phi_X(m)\right\}\exp\left\{\frac{-i\mu t^2}{\sigma^2 m}\right\} \\
&= \lim_{m\to 0} \exp\left\{\frac{t^2}{\sigma^2 m^2}\log\phi_X(m) - \frac{i\mu t^2}{\sigma^2 m}\right\} \\
&= \lim_{m\to 0} \exp\left\{-\frac{1}{2}t^2\left[\frac{2i\mu}{\sigma^2 m} - \frac{2}{\sigma^2 m^2}\log\phi_X(m)\right]\right\} \\
&= \exp\left\{-\frac{1}{2}t^2 \lim_{m\to 0}\frac{2i\mu m - 2\log\phi_X(m)}{\sigma^2 m^2}\right\}
\end{aligned}$$

To finish the proof, all we need to do is to show that limit in the last expression is equal to one. Notice that $\phi_X(0) = 1$, so the limit has the indeterminate form $\frac{0}{0}$. We can apply L'Hôpital's Rule:

$$\lim_{m\to 0}\frac{2i\mu m - 2\log\phi_X(m)}{\sigma^2 m^2} = \lim_{m\to 0}\frac{2i\mu - 2\frac{\phi_X'(m)}{\phi_X(m)}}{2\sigma^2 m}$$

but $\phi_X'(0) = i\,\mathbb{E}\,X = i\mu$, so we apply L'Hôpital's Rule again

$$\begin{aligned}
&= \lim_{m\to 0}\frac{-\left[\phi_X''(m)\phi_X(m) - \phi_X'(m)\phi_X'(m)\right]}{\sigma^2\phi_X(m)\phi_X(m)} \\
&= \frac{-\left[i^2\,\mathbb{E}\,X^2 - i^2(\mathbb{E}\,X)^2\right]}{\sigma^2} \\
&= -i^2\,\frac{\mathbb{E}\,X^2 - (\mathbb{E}\,X)^2}{\sigma^2} \\
&= 1
\end{aligned}$$