A short example on Bellman equations

Randall S. Romero Aguilar

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Introduction

This note shows the intuition behind the use of dynamic programming in the solution of dynamic programming problems. We present two models of a consumer who wants to maximize his lifetime consumption over an infinite horizon, by optimally allocating his resources through time. In the first model, the consumer uses a financial instrument (say a bank deposit without overdraft limit) to smooth consumption; in the second, the consumer has access to a production technology and uses the level of capital to smooth consumption.

To keep matters simple, we assume that there is a representative consumer whose instant utility function is logarithmic, and that there is no uncertainty in the model.

1 Consumption and financial assets

In this first model, the consumer is endowed with A units of the consumption good, but he does not have income. Savings can be kept in the form of a bank deposit, which yields a interest rate r.

The lifetime utility of the consumer is $U(c_0, c_1, \ldots, c_T) = \sum_{t=0}^{T} \beta^t \ln c_t$. His initial assets are A_0. The budget constraint is $A_{t+1} = (1 + r)(A_t - c_t)$. The consumer needs to choose the optimal values c_t^* that will maximize U. Once he chooses the sequence $\{c_t^*\}_{t=0}^T$ of optimal consumption, the maximum utility that he can achieved is ultimately constraint only by his initial assets A_0 and by how many periods he is going to live $T+1$. So define the *value function* V as the maximum utility the consumer can get as a function of his initial assets

$$V_{T+1}(A_0) = \max U(c_t) = \sum_{t=0}^{T} \beta^t \ln c_t^* \tag{1}$$

Consumer problem:

$$V_{T+1}(A_0) = \max_{\{c,A\}} \sum_{t=0}^{T} \beta^t \ln c_t, \qquad \text{(objective)} \tag{2a}$$

$$A_{t+1} = (1+r)(A_t - c_t), \qquad \text{(budget constraint)} \tag{2b}$$

$$A_{T+1} \geq 0 \qquad \text{(leave no debts)} \tag{2c}$$

We now solve the problem for special cases $T=0$, $T=1$, $T=2$, and then generalize for $T=\infty$.

Solution when T=0

In this case, consumer problem is simply

$$V_1(A_0) = max\,\{\ln c_0\} \text{ subject to}$$

$$A_1 = (1+r)(A_0 - c_0),$$

$$A_1 \geq 0$$

We need to find c_0 and A_1. Substitute $c_0 = A_0 - \frac{A_1}{1+r}$ in the objective function:

$$\max_{A_1} \ln\left[A_0 - \frac{A_1}{1+r}\right] \quad \text{subject to} \quad A_1 \geq 0$$

This function is strictly decreasing on A_1, so we set A_1 to its minimum possible value; given the constraint 2c, we set $A_1 = 0$, which implies that $c_0 = A_0$ and $V_1(A_0) = \ln A_0$. In words, in the last period of this life, it is optimal for a consumer to spend his entire assets.

Solution when T=1

The problem is now

$$
\begin{aligned}
V_2(A_0) = \{\ln c_0 + \beta \ln c_1\} \text{ subject to}\\
A_1 = (1+r)(A_0 - c_0),\\
A_2 = (1+r)(A_1 - c_1),\\
A_2 \geq 0
\end{aligned}
$$

We now need to find $c_0,\ c_1,\ A_1$ and A_2. Instead of solving today for all these quantities, think of solving today only for c_0 and A_1, and next period solving for the remaining c_1 and A_2. But from our example with $T = 0$ we learned that a consumer will spend his entire assets in the last period, so we know that in next period the consumer will set $c_1 = A_1$ (his remaining assets, which he will choose in the *current* period) and $A_2 = 0$. So we can rewrite the problem as

$$
\begin{aligned}
V_2(A_0) &= \max_{c_0,c_1,A_1,A_2} \{\ln c_0 + \beta \ln c_1\}\\
&= \max_{c_0,A_1} \left\{ \ln c_0 + \beta \max_{c_1,A_2} [\ln c_1] \right\}\\
&= \max_{c_0,A_1} \{\ln c_0 + \beta V_1(A_1)\}
\end{aligned}
$$

subject to $A_1 = (1+r)(A_0 - c_0)$. Again, we substitute $c_0 = A_0 - \frac{A_1}{1+r}$ and solve the problem

$$
\max_{A_1} \left\{ \ln \left[A_0 - \frac{A_1}{1+r} \right] + \beta V_1(A_1) \right\}
$$

The first order condition is

$$\frac{1}{c_0}\frac{-1}{1+r} + \beta V_1'(A_1) = 0 \Rightarrow 1 = (1+r)\beta c_0 V_1'(A_1)$$

From the example with $T = 0$ we know that $V_1(A) = \ln A$, then $V_1'(A_1) = \frac{1}{A_1}$. Substitute in the first order condition

$$1 = (1+r)\beta c_0 \tfrac{1}{A_1} \Rightarrow A_1^* = (1+r)\beta c_0^*$$

Now substitute in the budget constraint to get $(1+r)\beta c_0^* = (1+r)(A_0 - c_0^*)$. It follows that

$$c_0^* = \tfrac{1}{1+\beta}A_0 \Rightarrow A_1^* = \tfrac{(1+r)\beta}{1+\beta}A_0$$

and the value function is

$$\begin{aligned}
V_2(A_0) &= \ln c_0^* + \beta V_1(A_1^*) \\
&= \ln c_0^* + \beta \ln A_1^* \\
&= \ln c_0^* + \beta \ln[(1+r)\beta c_0^*] \\
&= (1+\beta)\ln c_0^* + \beta \ln \beta + \beta \ln(1+r) \\
&= (1+\beta)\ln A_0 - (1+\beta)\ln(1+\beta) + \beta \ln \beta + \beta \ln(1+r)
\end{aligned}$$

For simplicity, we can write $V_2(A_0) = (1+\beta)\ln A_0 + k_2$, where the term k_2 is just a constant: $k_2 = \beta \ln(1+r) + \beta \ln \beta - (1+\beta)\ln(1+\beta)$

Solution when T=2

The problem is now

$$V_3(A_0) = max\left\{\ln c_0 + \beta \ln c_1 + \beta^2 \ln c_2\right\} \text{ subject to}$$

$$A_1 = (1+r)(A_0 - c_0),$$

$$A_2 = (1+r)(A_1 - c_1),$$

$$A_3 = (1+r)(A_2 - c_2),$$

$$A_3 \geq 0$$

We will follow the same strategy that we used in the case $T = 1$ and choose only c_0 and A_1 this period, and let the consumer choose c_1, c_2, A_2, A_3 next period.

$$\begin{aligned} V_3(A_0) &= \max_{c_0,c_1,c_2,A_1,A_2,A_3} \left\{\ln c_0 + \beta \ln c_1 + \beta^2 \ln c_2\right\} \\ &= \max_{c_0,A_1} \left\{\ln c_0 + \beta \max_{c_1,c_2,A_2,A_3} [\ln c_1 + \beta \ln c_2]\right\} \\ &= \max_{c_0,A_1} \{\ln c_0 + \beta V_2(A_1)\} \end{aligned}$$

Again, we substitute $c_0 = A_0 - \frac{A_1}{1+r}$ and solve the problem

$$\max_{A_1} \left\{\ln\left[A_0 - \frac{A_1}{1+r}\right] + \beta V_2(A_1)\right\}$$

The first order condition is now

$$\frac{1}{c_0}\frac{-1}{1+r} + \beta V_2'(A_1) = 0 \Rightarrow 1 = (1+r)\beta c_0 V_2'(A_1)$$

From the example with $T = 1$ we know that $V_2(A) = (1+\beta)\ln A + k_2$. It follows that $V_2'(A_1) = \frac{1+\beta}{A_1}$. Substitute in the first order condition

$$1 = (1+r)\beta c_0 \tfrac{1+\beta}{A_1} \Rightarrow A_1^* = (1+r)(\beta + \beta^2)c_0^*$$

Now substitute in the budget constraint to get $(1+\beta)(1+r)\beta c_0^* = (1+r)(A_0 - c_0^*)$. Then

$$c_0^* = \tfrac{1}{1+\beta+\beta^2} A_0 \Rightarrow A_1^* = \tfrac{(1+r)(\beta+\beta^2)}{1+\beta+\beta^2} A_0$$

and the value function is

$$\begin{aligned}
V_3(A_0) &= \ln c_0^* + \beta V_2(A_1^*) \\
&= \ln c_0^* + \beta[(1+\beta)\ln(A_1^*) + k_2] \\
&= \ln c_0^* + (\beta+\beta^2)\ln(A_1^*) + \beta k_2 \\
&= \ln c_0^* + (\beta+\beta^2)\ln[(1+r)(\beta+\beta^2)c_0^*] + \beta k_2 \\
&= (1+\beta+\beta^2)\ln c_0^* + (\beta+\beta^2)\ln(1+r) + (\beta+\beta^2)\ln(\beta+\beta^2) + \beta k_2 \\
&= (1+\beta+\beta^2)\ln A_0 + k_3
\end{aligned}$$

where $k_3 = (\beta+\beta^2)\ln(1+r) + (\beta+\beta^2)\ln\beta + (\beta+\beta^2)\ln(1+\beta) - (1+\beta+\beta^2)\ln(1+\beta+\beta^2) + \beta k_2$. After substituting k_2 and simplifying we get

$$k_3 = (\beta+2\beta^2)\ln(1+r) + (\beta+2\beta^2)\ln\beta - (1+\beta+\beta^2)\ln(1+\beta+\beta^2)$$

Solution when $T = \infty$

Writing the consumer problem as a recursive optimization problem simplifies the task of solving for the infinite horizon case. Let V_∞ denote the value function, then from the previous examples we can write

$$\begin{aligned}V_\infty(A_0) &= \max_{c_j, A_{j+1}} \left\{ \sum_{t=0}^{\infty} \beta^t \ln c_t \right\} \\ &= \max_{c_0, A_1} \{\ln c_0 + \beta V_{\infty-1}(A_1)\} \\ A_{t+1} &= (1+r)(A_t - c_t), \quad \forall t = 0, 1, \ldots\end{aligned}$$

For an infinitely lived consumer, next period he will still have an infinite horizon ahead of him, so we could expect that $V \equiv V_\infty = V_{\infty-1}$. This result, where the sequence $\{V_j\}_{j=1}^{\infty}$ converges to V, the fixed point of the Bellman equation, requires that $|\beta| < 1$. To gain some intuition, observe the sequence $\{V_1, V_2, V_3\} = \{1 \ln A + k_1, (1+\beta)\ln A + k_2, (1+\beta+\beta^2)\ln A + k_3\}$; one could speculate that the term V_n should take the form $V_{n+1} = (1+\beta+\ldots+\beta^n)\ln A + k_{n+1}$, but its first term has a geometric series that converges to $\frac{1}{1-\beta}$ if and only if $-1 < \beta < 1$.

Again, we substitute $c_0 = A_0 - \frac{A_1}{1+r}$ in the Bellman equation and solve the problem

$$V(A_0) = \max_{A_1} \left\{ \ln \left[A_0 - \frac{A_1}{1+r} \right] + \beta V(A_1) \right\} \tag{3}$$

The first order condition is now

$$\frac{1}{c_0}\frac{-1}{1+r} + \beta V'(A_1) = 0 \Rightarrow 1 = (1+r)\beta c_0 V'(A_1) \tag{4}$$

Now we have a complication: we need the derivative of $V(A_1)$, but we don't know the form of V. We are going to apply a result, known as the envelope condition, that says that we can take the derivative of $V(A_0)$ in equation 3 **pretending** that A_1 is *not* a function of A_0. Envelope condition is:

$$V'(A_0) = \frac{1}{c_0} \tag{5}$$

From 5, we infer that $V'(A_1) = \frac{1}{c_1}$ and substitute in 4:

$$1 = (1 + r)\beta\frac{c_0}{c_1} \tag{6}$$

Equation 6 is known as the Euler equation. This implies that $c_1 = (1+r)\beta c_0$, or more generally $c_t = [(1+r)\beta]^t c_0$. Since the consumer lifetime budget constraint is $A_0 = sum_{t=0}^{\infty}[(1+r)^{-t}c_t]$, we obtain:

$$\begin{aligned}
A_0 &= \sum_{t=0}^{\infty}[(1+r)^{-t}c_t] \\
&= \sum_{t=0}^{\infty}[(1+r)^{-t}(1+r)^t\beta^t c_0] \\
&= c_0\sum_{t=0}^{\infty}\beta^t \\
&= \frac{c_0}{1-\beta} \\
\Rightarrow c_0^* &= (1-\beta)A_0 \qquad (7)
\end{aligned}$$

Then $c_t^* = (1+r)^t \beta^t (1-\beta) A_0$, and the value function is

$$
\begin{aligned}
V(A_0) &= \sum_{t=0}^{\infty} \beta^t \ln[(1+r)^t \beta^t (1-\beta) A_0] \\
&= \sum_{t=0}^{\infty} \beta^t \ln A_0 + \sum_{t=0}^{\infty} \beta^t \ln[(1+r)^t \beta^t (1-\beta)] \\
&= \ln A_0 \sum_{t=0}^{\infty} \beta^t + \sum_{t=0}^{\infty} \beta^t [t \ln(1+r)\beta + \ln(1-\beta)] \\
&= \ln A_0 \frac{1}{1-\beta} + \sum_{t=0}^{\infty} \beta^t [t \ln(1+r)\beta + \ln(1-\beta)] \\
&= \frac{1}{1-\beta} \ln A_0 + \ln[(1+r)\beta] \sum_{t=0}^{\infty} t\beta^t + \ln(1-\beta) \sum_{t=0}^{\infty} \beta^t \\
&= \frac{1}{1-\beta} \ln A_0 + \ln[(1+r)\beta] \frac{\beta}{1-\beta} \sum_{t=1}^{\infty} t(1-\beta)\beta^{t-1} + \frac{\ln(1-\beta)}{1-\beta} \\
&= \frac{1}{1-\beta} \ln A_0 + \ln[(1+r)\beta] \frac{\beta}{(1-\beta)^2} + \frac{\ln(1-\beta)}{1-\beta} \\
&= \frac{1}{1-\beta} \ln A_0 + \frac{\beta \ln(1+r) + \beta \ln \beta + (1-\beta)\ln(1-\beta)}{(1-\beta)^2}
\end{aligned}
$$

In the second to last step, we used the fact that $\sum_{t=1}^{\infty} t(1-\beta)\beta^{t-1}$ is the expected value of a geometric random variable with parameter $1-\beta$.

The last point in our discussion is to justify the envelope condition: deriving $V(A_0)$ pretending that A_1^* did not depend on A_0. But we know it does, so write $A_1^* = h(A_0)$ for some function h. From the definition of the value function write:

$$
V(A_0) = \ln\left[A_0 - \frac{h(A_0)}{1+r}\right] + \beta V(h(A_0)) \tag{8}
$$

Take derivative and arrange terms:

$$
\begin{aligned}
V'(A_0) &= \tfrac{1}{c_0}\left[1 - \frac{h'(A_0)}{1+r}\right] + \beta V'(h(A_0))h'(A_0) \\
&= \tfrac{1}{c_0} + \left[\tfrac{-1}{c_0}\frac{1}{1+r} + \beta V'(A_1^*)\right] h'(A_0)
\end{aligned}
$$

but the term in square brackets must be zero from the first order condition 4.

2 Consumption and physical investment

We now assume that the consumer is endowed with k units of a good that can be used either for consumption or for the production of additional good[1]. We refer to "capital" to the part of the good that is used for future production, and assume that capital fully depreciates with the production process.

The lifetime utility of the consumer is again $U(c_0, c_1, \ldots, c_\infty) = \sum_{t=0}^{\infty} \beta^t \ln c_t$, and his initial endowment of capital is k_0. The production function is $y = Ak^\alpha$, where $A > 0$ and $0 < \alpha < 1$ are parameters. The budget constraint is $c_t + k_{t+1} = Ak_t^\alpha$.

In this case, the Bellman equation is

$$
V(k_0) = \max_{c_0, k_1} \{\ln c_0 + \beta V(k_1)\}
$$

Substitute the constraint $c_0 = Ak_0^\alpha - k_1$ in the Bellman equation. To simplify the notation, we will drop the time index and will use a prime (as in k') to denote 'next period" variables.

In this case, the Bellman equation is

$$
V(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta V(k')\} \tag{9}
$$

[1]Think of a farmer who can either eat corn now or sow it to get more corn next year.

We will solve this equation by *value function iteration*. Before that, we will see the intuition behind this procedure by looking at a simple case of finding a fixed point for a function in $\mathbb{R}$.

2.1 A note on convergence of contraction mappings

How do we solve the Bellman equation? First of all, notice that the Bellman equation involves a *functional*, where the unknown is the function $V(k)$.

To put the problem in perspective, consider finding a fixed point for the *function* $f(x) = 1 + 0.5x$, for $x \in \mathbb{R}$. It is easy to see that $x^* = 2$ is a fixed point:

$$x^* = f(x^*) = 1 + 0.5x^* \rightarrow 0.5x^* = 1 \rightarrow x^* = 2$$

Suppose we could not solve the equation $x = 1 + 0.5x$ directly. How could we find the fixed point then? Notice that $|f'(x)| = |0.5| < 1$, so f is a contraction (that is, for all x and y, the function satisfies $|f(x) - f(y)| < |x - y|$). Then, if we start from an arbitrary point, say x_0, and by iteration we form the succession $x_{j+1} = f(x_j)$, we will get that $\lim_{j \to \infty} x_j = x^*$. For example, pick $x_0 = 6$:

$$\begin{aligned}
x_0 &= 6 \\
x_1 &= f(x_0) = 1 + \tfrac{6}{2} = 4 \\
x_2 &= f(x_1) = 1 + \tfrac{4}{2} = 3 \\
x_3 &= f(x_2) = 1 + \tfrac{3}{2} = 2.5 \\
x_4 &= f(x_3) = 1 + \tfrac{2.5}{2} = 2.25 \\
&\vdots
\end{aligned}$$

If we keep iterating, we will get arbitrarily close to the solution $x^* = 2$.

2.2 Value function iteration

Unfortunately, in equation 9 we cannot solve for V directly. However, we know that the Bellman equation is a contraction mapping (as long as $|\beta| < 1$) that has a fixed point (its solution). Let's them apply the same strategy we used when looking for the fixed point of $f(x) = 1 + 0.5x$. That is, let's pick an initial guess ($V_0(k) = 0$ is a convenient one) and them iterate over the Bellman equation by[2]

$$V_{j+1}(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta V_j(k')\} \tag{10}$$

Starting from $V_0 = 0$, the problem 10 becomes:

$$V_1(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta \times 0\}$$

Since the objective is decreasing on k' and we have the restriction $k' \geq 0$, the solution is simply to set $k^{*'} = 0$. Then $c^* = Ak^\alpha$

$$\begin{aligned} V_1(k) &= \ln c^* + \beta \times 0 \\ &= \ln A + \alpha \ln k \end{aligned}$$

This completes our first iteration. Let's now find V_2 using again equation 10:

$$V_2(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta[\ln A + \alpha \ln k']\}$$

First order condition for the maximization problem is

$$\frac{1}{Ak^\alpha - k'} = \frac{\alpha\beta}{k'} \quad \Rightarrow \quad k^{*'} = \frac{\alpha\beta}{1 + \alpha\beta} Ak^\alpha = \left(1 - \tfrac{1}{1+\alpha\beta}\right) Ak^\alpha$$

[2]Notice that the subscript j in this problem refers to a specific iteration in our procedure, not to the horizon of the consumer (as in the first model).

Then consumption is $c^* = Ak^\alpha - k^{*'} = \left(\frac{1}{1+\alpha\beta}\right) Ak^\alpha$ and

$$
\begin{aligned}
V_2(k) &= \ln(c^*) + \beta \ln A + \alpha\beta \ln k^{*'} \\
&= \ln\left(Ak^\alpha\right) - \ln(1+\alpha\beta) + \beta\ln A + \alpha\beta \ln\left[\tfrac{\alpha\beta}{1+\alpha\beta} Ak^\alpha\right] \\
&= [\alpha\beta\ln(\alpha\beta) - (1+\alpha\beta)\ln(1+\alpha\beta)] + (1+\beta+\alpha\beta)\ln A + \alpha(1+\alpha\beta)\ln K
\end{aligned}
$$

This completes the second iteration. Let's have one more:

$$V_3(k) = \max_{k'} \left\{\ln(Ak^\alpha - k') + \beta[\phi + \alpha(1+\alpha\beta)\ln k']\right\}$$

where $\phi = [\alpha\beta\ln(\alpha\beta) - (1+\alpha\beta)\ln(1+\alpha\beta)] + (1+\beta+\alpha\beta)\ln A$. The first order condition is

$$\frac{1}{Ak^\alpha - k'} = \frac{\alpha\beta(1+\alpha\beta)}{k'} \quad \Rightarrow \quad k^{*'} = \frac{\alpha\beta + \alpha^2\beta^2}{1+\alpha\beta+\alpha^2\beta^2} Ak^\alpha = \left(1 - \frac{1}{1+\alpha\beta+\alpha^2\beta^2}\right) Ak^\alpha$$

Then consumption is $c^* = \frac{1}{1+\alpha\beta+\alpha^2\beta^2} Ak^\alpha$

You might be tired by now of iterating this function. Me too! So let's try to find some patterns (unless you really want to iterate to infinity). Table 1 summarizes the results for the consumption policy function.

j	c^*
1	$(1)^{-1}Ak^\alpha$
2	$(1+\alpha\beta)^{-1}Ak^\alpha$
3	$(1+\alpha\beta+\alpha^2\beta^2)^{-1}Ak^\alpha$

Table 1: Consumption after 3 iterations

From the pattern on this table, we could guess that after j iterations, the consumption policy would look like:

$$c_j^* = (1 + \alpha\beta + \ldots + \alpha^j\beta^j)^{-1} Ak^\alpha$$

But remember that to converge to the fixed point, we need to iterate to infinity: so now simply take the limit $j \to \infty$ of the consumption function: since $0 < \alpha\beta < 1$, the geometric series converges, and so

$$c^* = (1 - \alpha\beta) A k^\alpha \tag{11}$$

$$k^{*'} = \alpha\beta A k^\alpha \tag{12}$$

To get the value function, start from:

$$V(k) = \max_{k'} \{\ln(Ak^\alpha - k') + \beta V(k')\} \tag{13}$$

The first order condition is:

$$\frac{1}{c^*} = \beta \frac{\partial V(k^{*'})}{\partial k'} \quad \Rightarrow \quad \frac{\partial V(k^{*'})}{\partial k'} = \frac{1}{\beta c^*}$$

Combining equations 11 and 12 we get $c^* = \frac{1-\alpha\beta}{\alpha\beta} k^{*'}$, substitute in first order condition and solve the resulting differential equation:

$$\frac{\partial V(k^{*'})}{\partial k'} = \frac{\alpha}{1 - \alpha\beta} \frac{1}{k^{*'}}$$

$$V(k^{*'}) = \tfrac{\alpha}{1-\alpha\beta} \ln k^{*'} + \zeta$$

where ζ is an integration constant. So we know that the value function is $V(k) = \frac{\alpha}{1-\alpha\beta} \ln k + \zeta$, and all is missing is to determine the coefficient ζ:

$$
\begin{aligned}
V(k) &= \ln c^* + \beta V(k^{*'}) \\
\tfrac{\alpha}{1-\alpha\beta}\ln k + \zeta &= \ln c^* + \tfrac{\alpha\beta}{1-\alpha\beta}\ln k^{*'} + \beta\zeta \\
&= \ln c^* + \tfrac{\alpha\beta}{1-\alpha\beta}\ln\left[\tfrac{\alpha\beta}{1-\alpha\beta}c^*\right] + \beta\zeta \\
&= \tfrac{1}{1-\alpha\beta}\ln c^* + \tfrac{\alpha\beta}{1-\alpha\beta}\ln\left[\tfrac{\alpha\beta}{1-\alpha\beta}\right] + \beta\zeta \\
&= \tfrac{1}{1-\alpha\beta}\ln(1-\alpha\beta) + \tfrac{1}{1-\alpha\beta}\ln A + \tfrac{1}{1-\alpha\beta}\ln k^\alpha + \tfrac{\alpha\beta}{1-\alpha\beta}\ln\left[\tfrac{\alpha\beta}{1-\alpha\beta}\right] + \beta\zeta \\
&\Rightarrow \\
(1-\beta)\zeta &= \frac{\ln(1-\alpha\beta) + \ln A + \alpha\beta\ln\left[\tfrac{\alpha\beta}{1-\alpha\beta}\right]}{1-\alpha\beta} \\
&\Rightarrow \\
\zeta &= \frac{\ln A + \alpha\beta\ln(\alpha\beta) + (1-\alpha\beta)\ln(1-\alpha\beta)}{(1-\beta)(1-\alpha\beta)}
\end{aligned}
$$