

# Dynamic programming

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# 1. Introduction

# About this lecture

- ▶ We study how to use Bellman equations to solve dynamic programming problems.
- ▶ We consider a consumer who wants to maximize his lifetime consumption over an infinite horizon, by optimally allocating his resources through time. Two alternative models:
  1. the consumer uses a financial instrument (say a bank deposit without overdraft limit) to smooth consumption;
  2. the consumer has access to a production technology and uses the level of capital to smooth consumption.
- ▶ To keep matters simple, we assume:
  - ▶ a logarithmic instant utility function;
  - ▶ there is no uncertainty.
- ▶ To start, we review some math that we'll need later.

# Static optimization

- ▶ Optimization is a predominant theme in economic analysis.
- ▶ For this reason, the classical calculus methods of finding free and constrained extrema occupy an important place in the economist's everyday tool kit.
- ▶ Useful as they are, **such tools are applicable only to static optimization problems.**
- ▶ The solution sought in such problems usually consists of **a single optimal magnitude** for every choice variable.
- ▶ It does not call for a schedule of optimal sequential action.

- ▶ In contrast, a **dynamic optimization** problem poses the question of what is the optimal magnitude of a choice variable **in each period of time** within the planning period.
- ▶ It is even possible to consider an infinite planning horizon.
- ▶ The solution of a dynamic optimization problem would thus take the form of an **optimal time path for every choice variable**, detailing the best value of the variable today, tomorrow, and so forth, till the end of the planning period.

# Basic ingredients

A simple type of dynamic optimization problem would contain the following basic ingredients:

1. a given **initial point** and a given **terminal point**;
2. a set of **admissible paths** from the initial point to the terminal point;
3. a set of **path values** serving as performance indices (cost, profit, etc.) associated with the various paths; and
4. a specified objective—either to maximize or to minimize the path value or performance index by choosing the **optimal path**.

# Alternative approaches to dynamic optimization

To find the optimal path, there are three major approaches:

## Calculus of variations

Dating back to the late 17th century, it works about variations in the **state** path.

## Optimal control theory

The problem is viewed as having both a state and a **control** path, focusing on variations of the control path.

## Dynamic programming

Which embeds the control problem in a family of control problems, focusing on the optimal value of the problem (**value function**).



# Salient features of dynamic optimization problems

- ▶ Although dynamic optimization is mostly couched in terms of a sequence of time, it is also possible to envisage the planning horizon as a sequence of stages in an economic process.
- ▶ In that case, dynamic optimization can be viewed as a problem of multistage decision making.
- ▶ The distinguishing feature, however, remains the fact that the optimal solution would involve more than one single value for the choice variable.

## 2. Basics of dynamic programming

# The principle of optimality

The dynamic programming approach is based on the **principle of optimality** (Bellman, 1957)

*An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

# Why dynamic programming?

Dynamic programming is a very attractive method for solving dynamic optimization problems because

- ▶ it offers **backward induction**, a method that is particularly amenable to programmable computers, and
- ▶ it facilitates incorporating uncertainty in dynamic optimization models.

We now introduce basic ideas and methods of dynamic programming (Ljungqvist and Sargent 2004)

- ▶ basic elements of a recursive optimization problem
- ▶ the Bellman equation
- ▶ methods for solving the Bellman equation
- ▶ the Benveniste-Scheikman formula

# Sequential problems

- ▶ Let  $\beta \in (0, 1)$  be a discount factor.
- ▶ We want to choose an infinite sequence of “controls”  $\{x_t\}_{t=0}^{\infty}$  to maximize

$$\sum_{t=0}^{\infty} \beta^t r(s_t, x_t) \quad (1)$$

subject to  $s_{t+1} = g(s_t, x_t)$ , with  $s_0 \in \mathbb{R}$  given.

- ▶ We assume that  $r(s_t, x_t)$  is a concave function and that the set  $\{(s_{t+1}, s_t) : s_{t+1} \leq g(s_t, x_t), x_t \in \mathbb{R}\}$  is convex and compact.

Dynamic programming seeks a time-invariant **policy function**  $h$  mapping the **state**  $s_t$  into the **control**  $x_t$ , such that the sequence  $\{x_t\}_{t=0}^{\infty}$  generated by iterating the two functions

$$\begin{aligned}x_t &= h(s_t) \\s_{t+1} &= g(s_t, x_t)\end{aligned}$$

starting from initial condition  $s_0$  at  $t = 0$ , solves the original problem. A solution in the form of equations is said to be **recursive**.

To find the policy function  $h$  we need to know the **value function**  $V(s)$ , which expresses the optimal value of the original problem, starting from an arbitrary initial condition  $s \in S$ . Define

$$V(s_0) = \max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(s_t, x_t)$$

subject to  $s_{t+1} = g(s_t, x_t)$ , with  $s_0$  given.

We do not know  $V(s_0)$  until after we have solved the problem, but if we knew it the policy function  $h$  could be computed by solving for each  $s \in S$  the problem

$$\max_x \{r(s, x) + \beta V(s')\}, \quad \text{s.t. } s' = g(s, x) \quad (2)$$



Thus, we have exchanged the original problem of finding an infinite sequence of controls that maximizes expression (1) for the problem of finding the optimal value function  $V(s)$  and a function  $h$  that solves the continuum of maximum problems (2) —one maximum problem for each value of  $s$ .

The function  $V(s), h(s)$  are linked by the **Bellman equation**

$$V(s) = \max_x \{r(s, x) + \beta V[g(s, x)]\} \quad (3)$$

The maximizer of the RHS is a **policy function**  $h(s)$  that satisfies

$$V(s) = r[s, h(s)] + \beta V\{g[s, h(s)]\} \quad (4)$$

This is a **functional equation** to be solved for the pair of unknown functions  $V(s), h(s)$ .

## Some properties

Under various particular assumptions about  $r$  and  $g$ , it turns out that

1. The Bellman equation has a unique strictly concave solution.
2. This solution is approached in the limit as  $j \rightarrow \infty$  by iterations on

$$V_{j+1}(s) = \max_x \{r(s, x) + \beta V_j(s')\}, \text{ s.t. } s' = g(s, x), s \text{ given}$$

starting from any bounded and continuous initial  $V_0$ .

3. There is a unique and time-invariant optimal policy of the form  $x_t = h(s_t)$ , where  $h$  is chosen to maximize the RHS of the Bellman equation.
4. Off corners, the limiting value function  $V$  is differentiable.

Side note:

Banach Fixed-Point Theorem

- ▶ A real-valued function  $f$  on an interval (or, more generally, a convex set in vector space) is said to be **concave** if, for any  $x$  and  $y$  in the interval and for any  $t \in [0, 1]$ ,

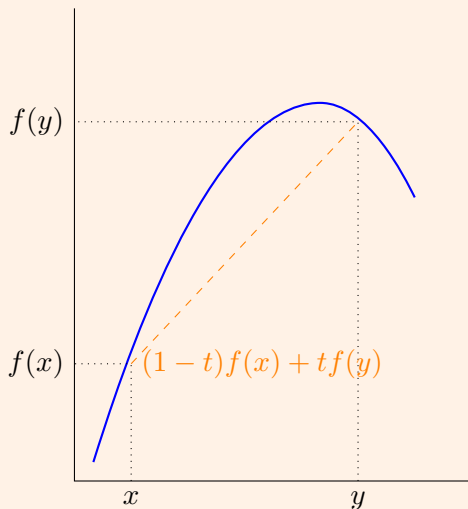
$$f((1-t)x + ty) \geq (1-t)f(x) + tf(y)$$

- ▶ A function is called **strictly concave** if

$$f((1-t)x + ty) > (1-t)f(x) + tf(y)$$

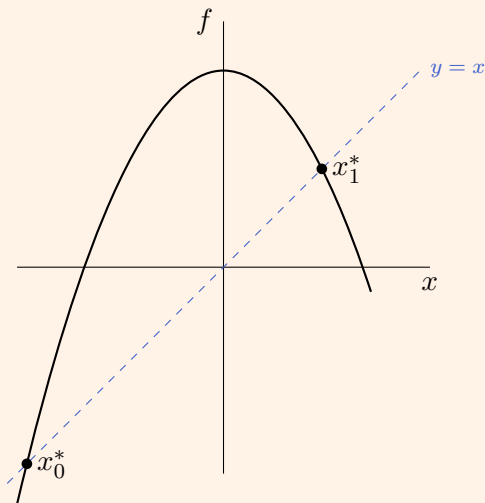
for any  $t \in (0, 1)$  and  $x \neq y$ .

For a function  $f : R \mapsto R$ , this definition merely states that for every  $z$  between  $x$  and  $y$ , the point  $(z, f(z))$  on the graph of  $f$  is above the straight line joining the points  $(x, f(x))$  and  $(y, f(y))$ .



# Fixed points

- ▶ A point  $x^*$  is a fixed-point of function  $f$  if it satisfies  $f(x^*) = x^*$ .
- ▶ Notice that  $f(f(\dots f(x^*) \dots)) = x^*$ .

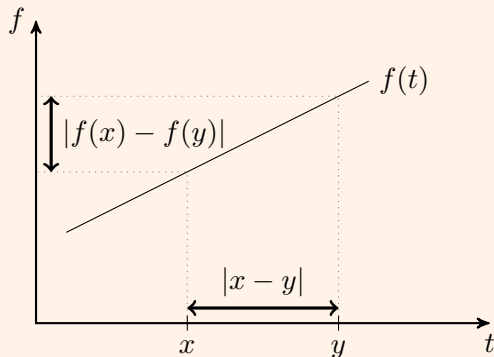


# Contraction mappings

A mapping  $f : X \mapsto X$  from a metric space  $X$  into itself is said to be a *strong contraction with modulus  $\delta$* , if  $0 \leq \delta < 1$  and

$$d(f(x), f(y)) \leq \delta d(x, y)$$

for all  $x$  and  $y$  in  $X$ .



# Banach Fixed-Point Theorem

If  $f$  is a strong contraction on a metric space  $X$ , then

- ▶ it possesses an unique fixed-point  $x^*$ , that is  $f(x^*) = x^*$
- ▶ if  $x_0 \in X$  and  $x_{i+1} = f(x_i)$ , then the  $x_i$  converge to  $x^*$

Proof: Use  $x_0$  and  $x^*$  in the definition of a strong contraction:

$$\begin{aligned}d(f(x_0), f(x_*)) &\leq \delta d(x_0, x^*) \Rightarrow \\d(x_1, x_*) &\leq \delta d(x_0, x^*) \Rightarrow \\d(x_k, x_*) &\leq \delta^k d(x_0, x^*) \rightarrow 0 \text{ as } k \rightarrow \infty\end{aligned}$$



Example 1:

Searching a fixed point by function iteration

- ▶ Consider finding a fixed point for the *function*  $f(x) = 1 + 0.5x$ , for  $x \in \mathbb{R}$ .
- ▶ It is easy to see that  $x^* = 2$  is a fixed point:

$$f(x^*) = f(2) = 1 + 0.5(2) = 2 = x^*$$

- ▶ Suppose we could not solve the equation  $x = 1 + 0.5x$  directly. How could we find the fixed point then?
- ▶ Notice that  $|f'(x)| = |0.5| < 1$ , so  $f$  is a contraction.

By Banach Theorem, if we start from an arbitrary point  $x_0$  and by iteration we form the sequence  $x_{j+1} = f(x_j)$ , it follows that  $\lim_{j \rightarrow \infty} x_j = x^*$ .

For example, pick:

$$x_0 = 6$$

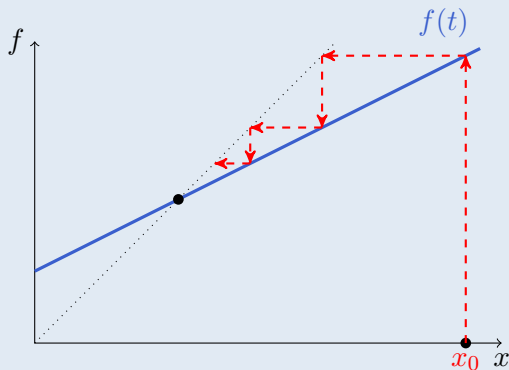
$$x_1 = f(x_0) = 1 + \frac{6}{2} = 4$$

$$x_2 = f(x_1) = 1 + \frac{4}{2} = 3$$

$$x_3 = f(x_2) = 1 + \frac{3}{2} = 2.5$$

$$x_4 = f(x_3) = 1 + \frac{2.5}{2} = 2.25$$

$\vdots$



If we keep iterating, we will get arbitrarily close to the solution  $x^* = 2$ .

Starting with the Bellman equation

$$V(s) = \max_x \{r(s, x) + \beta V[g(s, x)]\}$$

Since the value function is differentiable, the optimal  $x^* \equiv h(s)$  must satisfy the first-order condition

$$r_x(s, x^*) + \beta V'\{g(s, x^*)\}g_x(s, x^*) = 0 \quad (\text{FOC})$$

According to (4):  $V(s) = r[s, h(s)] + \beta V\{g[s, h(s)]\}$

If we also assume that the policy function  $h(s)$  is differentiable, differentiation of this expression yields

$$V'(s) = r_s[s, h(s)] + r_x[s, h(s)]h'(s) + \beta V'\{g[s, h(s)]\} \{g_s[s, h(s)] + g_x[s, h(s)]h'(s)\}$$

Arranging terms, substituting  $x^* = h(s)$  as the optimal policy

$$V'(s) = r_s(s, x^*) + \beta V'[g(s, x^*)]g_s(s, x^*) + \{r_x[s, x^*] + \beta V'\{g[s, x^*]\}g_x[s, x^*]\} h'(s)$$

The highlighted part cancels out because of (FOC), therefore

$$V'(s) = r_s(s, x^*) + \beta V'(s') g_s(s, x^*)$$

Notice that we could have obtained this result much faster by taking derivative of

$$V(s) = r(s, x^*) + \beta V[g(s, x^*)]$$

with respect to the state variable  $s$  **as if** the control variable  $x^* \equiv h(s)$  did not depend on  $s$ .

In the envelope condition

$$V'(s) = r_s(s, x^*) + \beta V'(s') g_s(s, x^*)$$

when the states and controls can be defined in such a way that only  $x$  appears in the transition equation, i.e.,

$$s' = g(x) \quad \Rightarrow \quad g_s(s, x^*) = 0,$$

the derivative of the value function becomes

$$V'(s) = r_s[s, h(s)] \tag{B-S}$$

This is a version of a formula of Benveniste and Scheinkman.

# Euler equations

- ▶ In many problems, there is no unique way of defining states and controls
- ▶ When the states and controls can be defined in such a way that  $s' = g(x)$ , the (FOC) for the Bellman equation together with the (B-S) formula implies

$$r_x(s_t, x_t) + \beta r_s(s_{t+1}, x_{t+1})g'(x_t) = 0$$

- ▶ This equation is called an **Euler equation**.
- ▶ If we can write  $x_t$  as a function of  $s_{t+1}$ , we can use it to eliminate  $x_t$  from the Euler equation to produce a second-order difference equation in  $s_t$ .



# Solving the Bellman equation

- ▶ In those cases in which we want to go beyond the Euler equation to obtain an explicit solution, we need to find the solution  $V$  of the Bellman equation (3)
- ▶ Given  $V$ , it is straightforward to solve (3) successively to compute the optimal policy.
- ▶ However, for infinite-horizon problems, we cannot use backward iteration.

# Three computational methods

- ▶ There are three main types of computational methods for solving dynamic programs. All aim to solve the Bellman equation
  - ▶ Guess and verify
  - ▶ Value function iteration
  - ▶ Policy function iteration
- ▶ Each method is easier said than done: it is typically impossible analytically to compute even *one* iteration.
- ▶ Usually we need computational methods for approximating solutions: pencil and paper are insufficient.

Example 2:

Computer solution of DP models

There are several computer programs available for solving dynamic programming models:

- ▶ The [CompEcon toolbox](#), a MATLAB toolbox accompanying Miranda and Fackler (2002) textbook.
- ▶ The [PyCompEcon toolbox](#), my (still incomplete) Python version of Miranda and Fackler toolbox.
- ▶ Additional examples are available at [quant-econ](#), a website by Sargent and Stachurski with Python and Julia scripts.

- ▶ This method involves guessing and verifying a solution  $V$  to the Bellman equation.
- ▶ It relies on the uniqueness of the solution to the equation
- ▶ because it relies on luck in making a good guess, it is not generally available.

# Value function iteration

- ▶ This method proceeds by constructing a sequence of value functions and associated policy functions.
- ▶ The sequence is created by iterating on the following equation, starting from  $V_0 = 0$ , and continuing until  $V_j$  has converged:

$$V_{j+1}(s) = \max_x \{r(s, x) + \beta V_j[g(s, x)]\}$$

# Policy function iteration

This method, also known as *Howard's improvement algorithm*, consists of the following steps:

1. Pick a feasible policy,  $x = h_0(s)$ , and compute the value associated with operating forever with that policy:

$$V_{h_j}(s) = \sum_{t=0}^{\infty} \beta^t r[s_t, h_j(s_t)]$$

where  $s_{t+1} = g[s_t, h_j(s_t)]$ , with  $j = 0$ .

2. Generate a new policy  $x = h_{j+1}(s)$  that solves the two-period problem

$$\max_x \{r(s, x) + \beta V_{h_j}[g(s, x)]\}$$

for each  $s$ .

3. Iterate over  $j$  to convergence on steps 1 and 2.

- ▶ We modify the transition equation and consider the problem of maximizing

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t r(s_t, x_t) \quad \text{s.t. } s_{t+1} = g(s_t, x_t, \epsilon_{t+1}) \quad (5)$$

with  $s_0$  given at  $t = 0$

- ▶  $\epsilon_t$  is a sequence of i.i.d. r.v. :  $\mathbb{P}[\epsilon_t \leq e] = F(e)$  for all  $t$
- ▶  $\epsilon_{t+1}$  is realized at  $t + 1$ , after  $x_t$  has been chosen at  $t$ .
- ▶ At time  $t$ :
  - ▶  $s_t$  is known
  - ▶  $s_{t+j}$  is unknown ( $j \geq 1$ )



- ▶ The problem is to choose a *policy* or **contingency plan**  $x_t = h(s_t)$ . The Bellman equation is

$$V(s) = \max_x \{r(s, x) + \beta \mathbb{E}[V(s') \mid s]\}$$

- ▶ where  $s' = g(s, x, \epsilon)$ ,
- ▶ and  $E\{V(s') \mid s\} = \int V(s') dF(\epsilon)$
- ▶ The solution  $V(s)$  of the B.E. can be computed by **value function iteration**.

- ▶ The FOC for the problem is

$$r_x(s, x) + \beta \mathbb{E} \{ V' (s') g_x(s, x, \epsilon) \mid s \} = 0$$

- ▶ When the states and controls can be defined in such a way that  $s$  does not appear in the transition equation,

$$V'(s) = r_s[s, h(s)]$$

- ▶ Substituting this formula into the FOC gives the **stochastic Euler equation**

$$r_x(s, x) + \beta \mathbb{E} \{ r_s(s', x') g_x(s, x, \epsilon) \mid s \} = 0$$

### 3. Consumption and financial assets: infinite horizon

To illustrate how dynamic programming works, we consider a intertemporal consumption problem.

- ▶ Planning horizon: infinite
- ▶ Instant utility depends on current consumption:  $u(c_t)$
- ▶ Constant utility discount rate  $\beta \in (0, 1)$
- ▶ Lifetime utility is:

$$U(c_0, c_1, \dots) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- ▶ The problem: choosing the optimal **sequence** of values  $\{c_t^*\}$  that will maximize  $U$ , subject to a budget constraint.

# A savings model

The consumer

- ▶ is endowed with  $A_0$  units of the consumption good,
- ▶ does not have income
- ▶ can save in a bank deposit, which yields a interest rate  $r$ .

The budget constraint is

$$A_{t+1} = R(A_t - c_t)$$

where  $R \equiv 1 + r$  is the *gross* interest rate.

# The value function

- ▶ Once he chooses the sequence  $\{c_t^*\}_{t=0}^{\infty}$  of optimal consumption, the maximum utility that he can achieved is ultimately constraint only by his initial assets  $A_0$ .
- ▶ So define the **value function**  $V$  as the maximum utility the consumer can get as a function of his initial assets

$$V(A_0) = \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to } A_{t+1} = R(A_t - c_t)$$

## Consumer problem:

$$V(A_0) = \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (\text{objective})$$

$$A_{t+1} = R(A_t - c_t) \quad \forall t = 0, 1, 2, \dots$$

(budget constraint)



# Dealing with the intertemporal budget constraint

Notice that we have a budget constraint for every time period  $t$ . So we form the Lagrangean

$$\begin{aligned} V(A_0) &= \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [R(A_t - c_t) - A_{t+1}] \\ &= \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left\{ \beta^t u(c_t) + \lambda_t [R(A_t - c_t) - A_{t+1}] \right\} \end{aligned}$$

Instead of dealing with the constraints explicitly, we can just substitute  $c_t = A_t - A_{t+1}/R$  in all time periods:

$$= \max_{\{A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left\{ \beta^t u \left( A_t - \frac{A_{t+1}}{R} \right) \right\}$$

So, we choose consumption implicitly by choosing the path of assets.

# A recursive approach to solving the problem

Keeping in mind that  $c_t = A_t - A_{t+1}/R$

$$\begin{aligned} V(A_0) &= \max_{\{A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &= \max_{\{A_{t+1}\}_{t=0}^{\infty}} \left\{ u(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_t) \right\} \\ &= \max_{\{A_{t+1}\}_{t=0}^{\infty}} \left\{ u(c_0) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right\} \end{aligned}$$

“An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”

$$\begin{aligned} &= \max_{A_1} \left\{ u(c_0) + \beta \max_{\{A_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{t+1}) \right\} \\ &= \max_{A_1} \{ u(c_0) + \beta V(A_1) \} \end{aligned}$$

## Bellman equation

$$V(A) = \max_{c, A'} \{u(c) + \beta V(A') + \lambda[R(A - c) - A']\}$$

- ▶ This says that the maximum lifetime utility the consumer can get must be equal to the sum of current utility plus the discounted value of the lifetime utility he will get starting next period.

# Three ways to write the Bellman equation

- ▶ Explicitly writing down the budget constraint (BC):

$$V(A) = \max_{c, A'} \{u(c) + \beta V(A') + \lambda[R(A - c) - A']\}$$

- ▶ Using the BC to substitute future assets:

$$V(A) = \max_c \{u(c) + \beta V[R(A - c)]\}$$

- ▶ Using the BC to substitute consumption:

$$V(A) = \max_{A'} \left\{ u \left( A - \frac{A'}{R} \right) + \beta V(A') \right\}$$

# Obtaining the Euler equation

The problem is

$$V(A) = \max_{A'} \left\{ u \left( A - \frac{A'}{R} \right) + \beta V(A') \right\}$$

so the FOCs is

$$-\frac{u'(c)}{R} + \beta V'(A') = 0 \quad \Rightarrow \quad u'(c) = \beta R V'(A')$$

The envelope condition is:

$$V'(A) = u'(c), \text{ which implies that } V'(A') = u'(c')$$

Substituting into the FOC we get the:

**Euler equation**

$$u'(c) = \beta R u'(c')$$

## Obtaining the Euler equation, second way

The Lagrangian for this problem is

$$V(A) = \max_{c, A'} \{u(c) + \beta V(A') + \lambda[R(A - c) - A']\}$$

so the FOCs are

$$\left. \begin{array}{l} u'(c) = \lambda R \\ \beta V'(A') = \lambda \end{array} \right\} \Rightarrow u'(c) = \beta R V'(A')$$

and the envelope condition is

$$V'(A) = \lambda R = u'(c)$$

which implies that

$$V'(A') = u'(c') \quad \Rightarrow \quad u'(c) = \beta R u'(c')$$

## (Not quite) obtaining the Euler equation

The problem is

$$V(A) = \max_c \{u(c) + \beta V[R(A - c)]\}$$

so the FOCs is

$$u'(c) - \beta R V'(A') = 0$$

but in this case the envelope condition is not useful:

$$V'(A) = \beta R V'(A')$$

## Euler equation

$$u'(c) = \beta R u'(c')$$

- ▶ This says that at the optimum, if the consumer gets one more unit of the good, he must be indifferent between consuming it now (getting  $u'(c)$ ) or saving it (which increases next-period assets by  $R$ ) and consuming it later, getting a discounted value of  $\beta R u'(c')$ .
- ▶ Notice that this is the same result we found on Lecture 8 (Applications of consumer theory), in the two-period intertemporal consumption problem!



Notice that the Euler equation can be written

$$u' \left( A_t - \frac{A_{t+1}}{R} \right) = \beta R u' \left( A_{t+1} - \frac{A_{t+2}}{R} \right)$$

which is a second-order nonlinear difference equation. In principle, it can be solved to obtain the

## Policy function

$$c_t^* = h(A_t)$$

consumption function

$$A_{t+1} = R[A_t - h(A_t)]$$

asset accumulation

## 4. Consumption and financial assets: finite horizon

- ▶ Planning horizon:  $T$  (possibly infinite)
- ▶ Instant utility depends on current consumption:  $u(c_t) = \ln c_t$
- ▶ Constant utility discount rate  $\beta \in (0, 1)$
- ▶ Lifetime utility is:

$$U(c_0, c_1, \dots, c_T) = \sum_{t=0}^T \beta^t \ln c_t$$

- ▶ The problem: choosing the optimal values  $c_t^*$  that will maximize  $U$ , subject to a budget constraint.

# A savings model

In this first model, the consumer

- ▶ is endowed with  $A_0$  units of the consumption good,
- ▶ does not have income
- ▶ can save in a bank deposit, which yields a interest rate  $r$ .

The budget constraint is

$$A_{t+1} = R(A_t - c_t)$$

where  $R \equiv 1 + r$  is the *gross* interest rate.

# The value function

- ▶ Once he chooses the sequence  $\{c_t^*\}_{t=0}^T$  of optimal consumption, the maximum utility that he can achieved is ultimately constraint only by his initial assets  $A_0$  and by how many periods he lives  $T + 1$ .
- ▶ So define the **value function**  $V$  as the maximum utility the consumer can get as a function of his initial assets

$$V_0(A_0) = \max_{\{c_t\}} \sum_{t=0}^T \beta^t \ln c_t^*$$

$$\text{subject to } A_{t+1} = R(A_t - c_t)$$

## Consumer problem:

$$V_0(A_0) = \max_{\{c, A\}} \sum_{t=0}^T \beta^t \ln c_t \quad (\text{objective})$$

$$A_{t+1} = R(A_t - c_t) \quad \forall t = 0, \dots, T$$

(budget constraint)

$$A_{T+1} \geq 0 \quad (\text{leave no debts})$$

## A time $\tau$ Bellman equation

Since the consumer problem is recursive, consider the value function for time  $t = \tau$

$$\begin{aligned} V_{\tau}(A_{\tau}) &= \max_{\{A_{t+1}\}_{t=\tau}^T} \sum_{t=\tau}^T \beta^{t-\tau} \ln c_t \\ &= \max_{\{A_{t+1}\}_{t=\tau}^T} \left\{ \ln c_{\tau} + \sum_{t=\tau+1}^T \beta^{t-\tau} \ln c_t \right\} \\ &= \max_{\{A_{t+1}\}_{t=\tau}^T} \left\{ \ln c_{\tau} + \beta \sum_{t=\tau+1}^T \beta^{t-(\tau+1)} \ln c_t \right\} \end{aligned}$$

Using Bellman optimality condition

$$\begin{aligned} &= \max_{A_{\tau+1}} \left\{ \ln c_{\tau} + \beta \max_{\{A_{t+1}\}_{t=\tau+1}^T} \sum_{t=\tau+1}^T \beta^{t-(\tau+1)} \ln c_t \right\} \\ &= \max_{A_{\tau+1}} \left\{ \ln c_{\tau} + \beta V_{\tau+1}(A_{\tau+1}) \right\} \end{aligned}$$

## Solving the time $\tau$ Bellman equation

With finite time horizon, the value function of one period depends on the value function for next period:

$$V_{\tau}(A_{\tau}) = \max_{A_{\tau+1}} \{ \ln c_{\tau} + \beta V_{\tau+1}(A_{\tau+1}) \}$$

Keeping in mind that  $c_{\tau} = A_{\tau} - A_{\tau+1}/R$ , the FOC is

$$\frac{-1}{Rc_{\tau}} + \beta V'_{\tau+1}(A_{\tau+1}) = 0 \quad \Rightarrow \quad 1 = R\beta c_{\tau} V'_{\tau+1}(A_{\tau+1})$$

So this problem can be solved by:

### Time $\tau$ solution:

$$\begin{aligned} 1 &= R\beta c_{\tau} V'_{\tau+1}(A_{\tau+1}) && \text{(first-order condition)} \\ A_{\tau+1} &= R(A_{\tau} - c_{\tau}) && \text{(budget constraint)} \end{aligned}$$

We now solve the problem for special cases  $t = T$ ,  $t = T - 1$ ,

$t = T - 2$ . Then we generalize for  $T = \infty$ .



## Solution when $t = T$

In this case, consumer problem is simply

$$V_T(A_T) = \max_{c_T, A_{T+1}} \{\ln c_T\} \text{ subject to}$$
$$A_{T+1} = R(A_T - c_T), \quad A_{T+1} \geq 0$$

We need to find  $c_T$  and  $A_{T+1}$ . Substitute  $c_T = A_T - \frac{A_{T+1}}{R}$  in the objective function:

$$\max_{A_{T+1}} \ln \left[ A_T - \frac{A_{T+1}}{R} \right] \text{ subject to } A_{T+1} \geq 0$$

This function is strictly decreasing on  $A_{T+1}$ , so we set  $A_{T+1}$  to its minimum possible value; given the transversality constraint we set  $A_{T+1} = 0$ , which implies  $c_T = A_T$  and  $V_T(A_T) = \ln A_T$ . In words, in his last period a consumer spends his entire assets.

## Solution when $t = T - 1$

The problem is now

$$V_{T-1}(A_{T-1}) = \max_{A_T} \{\ln c_{T-1} + \beta V_T(A_T)\}$$

Its solution, since we know that  $V_T(A_T) = \ln A_T$ , is given by:

$$\begin{cases} 1 = R\beta c_{T-1} V_T'(A_T) \\ A_T = R(A_{T-1} - c_{T-1}) \end{cases} \Rightarrow \begin{cases} A_T = R\beta c_{T-1} \\ A_T = R(A_{T-1} - c_{T-1}) \end{cases}$$

It follows that

$$c_{T-1}^* = \frac{1}{1+\beta} A_{T-1} \Rightarrow A_T^* = \frac{R\beta}{1+\beta} A_{T-1}$$

The value function is

$$\begin{aligned}V_{T-1}(A_{T-1}) &= \ln c_{T-1}^* + \beta V_T(A_T^*) \\ &= \ln c_{T-1}^* + \beta \ln A_T^* \\ &= \ln c_{T-1}^* + \beta \ln[R\beta c_{T-1}^*] \\ &= (1 + \beta) \ln c_{T-1}^* + \beta \ln \beta + \beta \ln R \\ &= (1 + \beta) \ln A_{T-1} - (1 + \beta) \ln(1 + \beta) + \dots \\ &\quad \dots + \beta \ln \beta + \beta \ln R \\ &= (1 + \beta) \ln A_{T-1} + \theta_{T-1}\end{aligned}$$

where the term  $\theta_{T-1}$  is just a constant.

## Solution when $t = T - 2$

The problem is now

$$V_{T-2}(A_{T-2}) = \max_{A_{T-1}} \{ \ln c_{T-2} + \beta V_{T-1}(A_{T-1}) \}$$

Its solution, since we know that

$V_{T-1}(A_{T-1}) = (1 + \beta) \ln A_{T-1} + \theta_{T-1}$ , is given by:

$$\begin{cases} 1 = R\beta c_{T-2} V'_{T-1}(A_{T-1}) \\ A_{T-1} = R(A_{T-2} - c_{T-2}) \end{cases} \Rightarrow \begin{cases} A_{T-1} = R\beta(1 + \beta)c_{T-2} \\ A_{T-1} = R(A_{T-2} - c_{T-2}) \end{cases}$$

It follows that

$$c_{T-2}^* = \frac{1}{1 + \beta + \beta^2} A_{T-2} \quad \Rightarrow \quad A_{T-1}^* = \frac{R(\beta + \beta^2)}{1 + \beta + \beta^2} A_{T-2}$$

The value function is

$$\begin{aligned}V_{T-2}(A_{T-2}) &= \ln c_{T-2}^* + \beta V_{T-1}(A_{T-1}^*) \\ &= \ln c_{T-2}^* + \beta[(1 + \beta) \ln(A_{T-1}^*) + \theta_{T-1}] \\ &= \ln c_{T-2}^* + (\beta + \beta^2) \ln[R(\beta + \beta^2)c_{T-2}^*] + \beta\theta_{T-1} \\ &= (1 + \beta + \beta^2) \ln c_{T-2}^* + (\beta + \beta^2)[\ln R + \dots \\ &\quad \dots + \ln(\beta + \beta^2)] + \beta\theta_{T-1} \\ &= (1 + \beta + \beta^2) \ln A_{T-2} + \theta_{T-2}\end{aligned}$$

where

$$\theta_{T-2} = (\beta + 2\beta^2) \ln R + (\beta + 2\beta^2) \ln \beta - (1 + \beta + \beta^2) \ln(1 + \beta + \beta^2)$$

If we keep iterating, the problem is now

$$V_{T-k}(A_{T-k}) = \max_{A_{T-k+1}} \{ \ln c_{T-k} + \beta V_{T-k+1}(A_{T-k+1}) \}$$

Its solution, is given by:

$$\begin{cases} 1 = R\beta c_{T-k} V'_{T-k+1}(A_{T-k+1}) \\ A_{T-k+1} = R(A_{T-k} - c_{T-k}) \end{cases}$$

But since we do not know  $V_{T-k+1}(A_{T-k+1})$ , we cannot substitute just yet, unless we solve for all intermediate steps. **Instead of doing that, we will search for patterns in our results.**

# Searching for patterns

Let's summarize the results for the policy function.

---

$t$	$c_t^*$	$A_{t+1}^*$
$T$	$A_T$	$0A_T$
$T-1$	$\frac{1}{1+\beta}A_{T-1}$	$R\beta\frac{1}{1+\beta}A_{T-1}$
$T-2$	$\frac{1}{1+\beta+\beta^2}A_{T-2}$	$R\beta\frac{1+\beta}{1+\beta+\beta^2}A_{T-2}$

---

We could guess that after  $k$  iterations:

$$\begin{aligned} T-k & \quad \frac{1}{1+\beta+\dots+\beta^k}A_{T-k} & \quad R\beta\frac{1+\beta+\dots+\beta^{k-1}}{1+\beta+\dots+\beta^k}A_{T-k} \\ & = & \quad \frac{1-\beta}{1-\beta^{k+1}}A_{T-k} & \quad R\beta\frac{1-\beta^k}{1-\beta^{k+1}}A_{T-k} \end{aligned}$$

---

# The time path of assets

Since  $A_{T-k+1} = R\beta \frac{1-\beta^k}{1-\beta^{k+1}} A_{T-k}$ , setting  $k = T, T - 1$ :

$$\begin{aligned} A_1 &= R\beta \frac{1 - \beta^T}{1 - \beta^{T+1}} A_0 \\ A_2 &= R\beta \frac{1 - \beta^{T-1}}{1 - \beta^T} A_1 \\ &= (R\beta)^2 \frac{1 - \beta^{T-1}}{1 - \beta^{T+1}} A_0 \end{aligned}$$

Iterating in this fashion we find that

$$A_t = (R\beta)^t \frac{1 - \beta^{T+1-t}}{1 - \beta^{T+1}} A_0$$



# The time path of consumption

Since  $c_{T-k}^* = \frac{1-\beta}{1-\beta^{k+1}} A_{T-k}$ , setting  $t = T - k$  we get consumption

$$\begin{aligned}c_t^* &= \frac{1-\beta}{1-\beta^{T+1-t}} A_t \\&= \frac{1-\beta}{1-\beta^{T+1-t}} \left[ (R\beta)^t \frac{1-\beta^{T+1-t}}{1-\beta^{T+1}} A_0 \right] \\&= (R\beta)^t \frac{1-\beta}{1-\beta^{T+1}} A_0 \\&\quad \phi\end{aligned}$$

That is

$$\ln c_t^* = t \ln(R\beta) + \ln \phi$$

# The time 0 value function

Substitution of the optimal consumption path in the Bellman equation give the value function

$$\begin{aligned}V_0(A_0) &\equiv \sum_{t=0}^T \beta^t \ln c_t^* = \sum_{t=0}^T \beta^t (t \ln(R\beta) + \ln \phi) \\&= \ln(R\beta) \sum_{t=0}^T \beta^t t + \ln \phi \sum_{t=0}^T \beta^t \\&= \frac{\beta}{1-\beta} \left( \frac{1-\beta^T}{1-\beta} - T\beta^T \right) \ln(R\beta) + \frac{1-\beta^{T+1}}{1-\beta} \ln \phi \\&= \frac{\beta}{1-\beta} \left( \frac{1-\beta^T}{1-\beta} - T\beta^T \right) \ln(R\beta) + \dots \\&\quad + \frac{1-\beta^{T+1}}{1-\beta} \ln \frac{1-\beta}{1-\beta^{T+1}} + \frac{1-\beta^{T+1}}{1-\beta} \ln A_0\end{aligned}$$

# From finite horizon to infinite horizon

Our results so far are

$$A_t = (R\beta)^t \frac{1 - \beta^{T+1-t}}{1 - \beta^{T+1}} A_0 \quad c_t^* = (R\beta)^t \frac{1 - \beta}{1 - \beta^{T+1}} A_0$$

$$V_0(A_0) = \frac{\beta}{1 - \beta} \left( \frac{1 - \beta^T}{1 - \beta} - T\beta^T \right) \ln(R\beta) + \frac{1 - \beta^{T+1}}{1 - \beta} \ln \frac{1 - \beta}{1 - \beta^{T+1}} + \frac{1 - \beta^{T+1}}{1 - \beta} \ln A_0$$

Taking the limit as  $T \rightarrow \infty$

$$A_t = (R\beta)^t A_0 \quad c_t^* = (R\beta)^t (1 - \beta) A_0 = (1 - \beta) A_t$$

$$V_0(A_0) = \frac{1}{1 - \beta} \ln A_0 + \frac{\beta \ln R + \beta \ln \beta + (1 - \beta) \ln(1 - \beta)}{(1 - \beta)^2}$$

## Policy function

$$c_t^* = (1 - \beta)A_t$$

consumption function

$$A_{t+1} = R\beta A_t$$

asset accumulation

- ▶ This says that the **optimal consumption rule** is, in every period, to consume a fraction  $1 - \beta$  of available initial assets.
- ▶ Over time, assets will increase, decrease or remain constant depending on how the degree of impatience  $\beta$  compares to reward to postpone consumption  $R$ .

# Time-variant value function

Now let's summarize the results for the value function:

---

$t$	$V_t(A)$
$T$	$\ln A$
$T - 1$	$(1 + \beta) \ln A + \theta_{T-1}$
$T - 2$	$(1 + \beta + \beta^2) \ln A + \theta_{T-2}$
	$\vdots$
$0$	$\frac{1}{1 - \beta} \ln A + \frac{\beta}{1 - \beta} \left( \frac{1 - \beta^T}{1 - \beta} - T\beta^T \right) \ln(R\beta) + \frac{1}{1 - \beta} \ln \frac{1 - \beta}{1 - \beta^{T+1}}$

---

Notice that the value function changes each period, **but only because each period the remaining horizon becomes one period shorter.**

# Time-invariant value function

Remember that in our  $k$  iteration,

$$V_{T-k}(A_{T-k}) = \max_{\substack{c_{T-k}, \\ A_{T-k+1}}} \{ \ln c_{T-k} + \beta V_{T-k+1}(A_{T-k+1}) \}$$

With an infinite horizon, the remaining horizon is the same in  $T-k$  and in  $T-k+1$ , so the value function is the same, precisely the **fixed-point of the Bellman equation**. Then we can write

$$V(A_{T-k}) = \max_{\substack{c_{T-k}, \\ A_{T-k+1}}} \{ \ln c_{T-k} + \beta V(A_{T-k+1}) \}$$

or simply

$$V(A) = \max_{c, A'} \{ \ln c + \beta V(A') \}$$

where a prime indicates a next-period variable

# The first order condition

Using the budget constraint to substitute consumption

$$V(A) = \max_{A'} \left\{ \ln \left( A - \frac{A'}{R} \right) + \beta V(A') \right\}$$

we obtain the FOC:

$$1 = R\beta c V'(A')$$

Despite not knowing  $V$ , we can determine its first derivative using the **envelope condition**. Thus, from

$$V(A) = \ln \left( A - \frac{A'^*}{R} \right) + \beta V(A'^*)$$

we get

$$V'(A) = \frac{1}{c}$$

- ▶ Because the solution is time-invariant,  $V'(A) = \frac{1}{c}$  implies that  $V'(A') = \frac{1}{c'}$ .
- ▶ Substitute this into the FOC to obtain the

## Euler equation

$$1 = R\beta \frac{c}{c'} = R\beta \frac{u'(c')}{u'(c)}$$

- ▶ This says that the marginal rate of substitution of consumption between any consecutive periods  $\frac{u'(c)}{\beta u'(c')}$  must equal the relative price of the later consumption in terms of the earlier consumption  $R$ .



# Value function iteration

- ▶ Suppose we wanted to solve the infinite horizon problem

$$V(A) = \max_{c, A'} \{ \ln c + \beta V(A') \} \text{ subject to } A' = R(A - c)$$

by value function iteration:

$$V_{j+1}(A) = \max_{c, A'} \{ \ln c + \beta V_j(A') \} \text{ subject to } A' = R(A - c)$$

- ▶ If we start iterating from  $V_0(A) = 0$ , our iterations would look identical to the procedure we used to solve for the finite horizon problem!

- ▶ Then, our iterations would look like

$j$	$V_j(A)$
0	0
1	$\ln A$
2	$(1 + \beta) \ln A + \theta_2$
3	$(1 + \beta + \beta^2) \ln A + \theta_3$
	$\vdots$

- ▶ If we keep iterating, we would expect that the coefficient on  $\ln A$  would converge to  $1 + \beta + \beta^2 + \dots = \frac{1}{1-\beta}$
- ▶ However, it is much harder to see a pattern on the  $\theta_j$  sequence.
- ▶ Then, we could try now the **guess and verify**, guessing that the solution takes the form  $V(A) = \frac{1}{1-\beta} \ln A + \theta$ .

- ▶ Our guess:  $V(A) = \frac{1}{1-\beta} \ln A + \theta$
- ▶ Solution must satisfy the FOC:  $1 = R\beta cV'(A')$  and budget constraint  $A' = R(A - c)$ .
- ▶ Combining these conditions we find  $c^* = (1 - \beta)A$  and  $A'^* = R\beta A$ .
- ▶ To be a solution of the Bellman equation, it must be the case that both sides are equal:

LHS	RHS
$V(A)$	$\ln c^* + \beta V(A'^*)$
$\frac{1}{1-\beta} \ln A + \theta$	$= \ln(1 - \beta)A + \beta \left[ \frac{\ln A'^*}{1-\beta} + \theta \right]$ $= \ln(1 - \beta)A + \beta \left[ \frac{\ln R\beta A}{1-\beta} + \theta \right]$ $= \frac{1}{1-\beta} \ln A + \frac{\beta}{1-\beta} \ln R\beta + \ln(1 - \beta) + \beta\theta$

The two sides are equal if and only if

$$\theta = \frac{\beta}{1-\beta} \ln R\beta + \ln(1 - \beta) + \beta\theta$$

That is, if

$$\theta = \frac{\beta \ln R + \beta \ln \beta + (1 - \beta) \ln(1 - \beta)}{(1 - \beta)^2}$$

## Why the envelope condition works?

The last point in our discussion is to justify the envelope condition: deriving  $V(A)$  pretending that  $A'^*$  did not depend on  $A$ . But we know it does, so write  $A'^* = h(A)$  for some function  $h$ . From the definition of the value function write:

$$V(A) = \ln \left[ A - \frac{h(A)}{R} \right] + \beta V(h(A))$$

Take derivative and arrange terms:

$$\begin{aligned} V'(A) &= \frac{1}{c} \left[ 1 - \frac{h'(A)}{R} \right] + \beta V'(h(A)) h'(A) \\ &= \frac{1}{c} + \left[ \frac{-1}{cR} + \beta V'(A'^*) \right] h'(A) \end{aligned}$$

but the term in square brackets must be zero from the FOC.

## 5. Consumption and physical investment

# A model with production

In this model

- ▶ the consumer is endowed with  $k_0$  units of a good that can be used either for consumption or for the production of additional good
- ▶ we refer to “capital” to the part of the good that is used for future production
- ▶ capital fully depreciates with the production process.
- ▶ The lifetime utility of the consumer is again
$$U(c_0, c_1, \dots, c_\infty) = \sum_{t=0}^{\infty} \beta^t \ln c_t,$$
- ▶ The production function is  $y = Ak^\alpha$ , where  $A > 0$  and  $0 < \alpha < 1$  are parameters.
- ▶ The budget constraint is  $c_t + k_{t+1} = Ak_t^\alpha$ .

## Consumer problem:

$$V(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t \quad (\text{objective})$$

$$k_{t+1} = Ak_t^\alpha - c_t \quad (\text{resource constraint})$$



# The Bellman equation

- ▶ In this case, the Bellman equation is

$$V(k_0) = \max_{c_0, k_1} \{ \ln c_0 + \beta V(k_1) \}$$

- ▶ Substitute the constraint  $c_0 = Ak_0^\alpha - k_1$  in the BE. To simplify notation, we drop the time index and use a prime (as in  $k'$ ) to denote “next period” variables. Then, BE is

$$V(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta V(k') \}$$

- ▶ We will solve this equation by **value function iteration**.

# The Euler equation

Remember that  $u(c) = \ln c$ ,  $y = f(k) = Ak^\alpha$ , so Bellman equation can be written as:

$$V(k) = \max_{k'} \{u(f(k) - k') + \beta V(k')\}$$

we get the FOC  $u'(c) = \beta V'(k')$  and the envelope condition  $V'(k) = u'(c)f'(k)$

## Euler equation

$$u'(c) = \beta f'(k')u'(c')$$

Notice how this result is similar to the one we got in the savings model: the return for giving up one unit of current consumption is:

**savings model:**  $R = 1 + r$ , the gross interest rate.

**physical capital model:**  $f'(k')$ , the marginal product of capital.

# Solving Bellman equation by function iteration

- ▶ How do we solve the Bellman equation?

$$V(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta V(k') \}$$

- ▶ This equation involves a *functional*, where the unknown is the function  $V(k)$ .
- ▶ Unfortunately, we cannot solve for  $V$  directly.
- ▶ However, this equation is a contraction mapping (as long as  $|\beta| < 1$ ) that has a fixed point (its solution).
- ▶ Let's pick an initial guess ( $V_0(k) = 0$  is a convenient one) and then iterate over the Bellman equation by\*

$$V_{j+1}(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta V_j(k') \}$$

Starting from  $V_0 = 0$ , the problem becomes:

$$V_1(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta \times 0 \}$$

Since the objective is decreasing on  $k'$  and we have the restriction  $k' \geq 0$ , the solution is simply  $k'^* = 0$ . Then  $c^* = Ak^\alpha$

$$\begin{aligned} V_1(k) &= \ln c^* + \beta \times 0 \\ &= \ln(Ak^\alpha) \\ &= \ln A + \alpha \ln k \end{aligned}$$

This completes our first iteration. Let's now find  $V_2$ :

$$V_2(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta[\ln A + \alpha \ln k'] \}$$

FOC is

$$\frac{1}{Ak^\alpha - k'} = \frac{\alpha\beta}{k'} \Rightarrow k'^* = \frac{\alpha\beta}{1 + \alpha\beta} Ak^\alpha = \theta_1 Ak^\alpha$$

Then consumption is  $c^* = (1 - \theta_1)Ak^\alpha = \frac{1}{1 + \alpha\beta} Ak^\alpha$  and

$$\begin{aligned} V_2(k) &= \ln(c^*) + \beta \ln A + \alpha\beta \ln k'^* \\ &= \ln(1 - \theta_1) + \ln(Ak^\alpha) + \beta[\ln A + \alpha \ln \theta_1 + \alpha \ln(Ak^\alpha)] \\ &= (1 + \alpha\beta) \ln(Ak^\alpha) + \beta \ln A + [\ln(1 - \theta_1) + \alpha\beta \ln \theta_1] \\ &= (1 + \alpha\beta) \ln(Ak^\alpha) + \phi_2 \end{aligned}$$

This completes the second iteration.

Let's have one more:

$$V_3(k) = \max_{k'} \{ \ln(Ak^\alpha - k') + \beta[(1 + \alpha\beta) \ln(Ak'^\alpha) + \phi_2] \}$$

The FOC is

$$\frac{1}{Ak^\alpha - k'} = \frac{\alpha\beta(1 + \alpha\beta)}{k'}$$
$$k'^* = \frac{\alpha\beta + \alpha^2\beta^2}{1 + \alpha\beta + \alpha^2\beta^2} Ak^\alpha = \theta_2 Ak^\alpha$$

Then consumption is  $c^* = (1 - \theta_2)Ak^\alpha = \frac{1}{1 + \alpha\beta + \alpha^2\beta^2} Ak^\alpha$

After substitution of  $c^*$  and  $k'^*$  into the Bellman equation (and a lot of cumbersome algebra):

$$V_3(k) = \ln(c^*) + \beta[(1 + \alpha\beta) \ln(Ak'^*\alpha) + \phi_2]$$
$$= (1 + \alpha\beta + \alpha^2\beta^2) \ln(Ak^\alpha) + \phi_3$$

This completes the third iteration.

You might be tired by now of iterating this function. Me too! So let's try to find some patterns (unless you really want to iterate to infinity). Let's summarize the results for the value function.

$j$	$V(k)$
1	$(1) \ln(Ak^\alpha)$
2	$(1 + \alpha\beta) \ln(Ak^\alpha) + \phi_2$
3	$(1 + \alpha\beta + \alpha^2\beta^2) \ln(Ak^\alpha) + \phi_3$

From this table, we could guess that after  $j$  iterations, the consumption policy would look like:

$$V_j(k) = (1 + \alpha\beta + \dots + \alpha^j\beta^j) \ln(Ak^\alpha) + \phi_j$$

- ▶ To converge to the fixed point, we need to iterate to infinity.
- ▶ Simply take the limit  $j \rightarrow \infty$  of the value function: since  $0 < \alpha\beta < 1$ , the geometric series converges, and so

$$V(k) = \frac{1}{1-\alpha\beta} \ln(Ak^\alpha) + \Phi$$

- ▶ Notice that we did not formally prove that the  $\phi_j$  sequence actually converges (so far we are just assuming it does.)



## Solving by guess and verify

- ▶ So far, we have not actually solved the Bellman equation, but the pattern we found allow us to **guess** that the value function is  $V(k) = \frac{1}{1-\alpha\beta} \ln(Ak^\alpha) + \Phi$ , where  $\Phi$  is an unknown coefficient.
- ▶ We are now going to **verify** that this function is the solution, finding the value of  $\Phi$
- ▶ This is called the **method of undetermined coefficients**.
- ▶ In this case, the Bellman equation is

$$V(k) = \max_{k'} \left\{ \ln(Ak^\alpha - k') + \frac{\beta}{1-\alpha\beta} \ln(Ak'^\alpha) + \beta\Phi \right\}$$

- ▶ FOC is

$$\frac{1}{Ak^\alpha - k'} = \frac{\alpha\beta}{(1 - \alpha\beta)k'} \quad \Rightarrow \quad \begin{cases} k'^* = \alpha\beta Ak^\alpha = \alpha\beta y \\ c^* = (1 - \alpha\beta)Ak^\alpha = (1 - \alpha\beta)y \end{cases}$$

Substitute in the Bellman equation is

$$\begin{aligned}V(k) &= \ln c^* + \frac{\beta}{1-\alpha\beta} \ln(Ak'^{\alpha}) + \beta\Phi \\ \frac{1}{1-\alpha\beta} \ln(Ak^{\alpha}) + \Phi &= \ln[(1-\alpha\beta)y] + \frac{\beta}{1-\alpha\beta} \ln[A(\alpha\beta y)^{\alpha}] + \beta\Phi \\ &= \left(1 + \frac{\alpha\beta}{1-\alpha\beta}\right) \ln y + \ln(1-\alpha\beta) + \frac{\beta \ln[A(\alpha\beta)^{\alpha}]}{1-\alpha\beta} + \beta\Phi \\ &= \frac{1}{1-\alpha\beta} \ln y + \frac{(1-\alpha\beta) \ln(1-\alpha\beta) + \beta \ln[A(\alpha\beta)^{\alpha}]}{1-\alpha\beta} + \beta\Phi\end{aligned}$$

Therefore

$$\begin{aligned}(1-\beta)\Phi &= \frac{(1-\alpha\beta) \ln(1-\alpha\beta) + \beta \ln[A(\alpha\beta)^{\alpha}]}{1-\alpha\beta} \\ \Phi &= \frac{\beta \ln A + \ln(\alpha\beta)^{\alpha\beta} + \ln(1-\alpha\beta)^{(1-\alpha\beta)}}{(1-\beta)(1-\alpha\beta)}\end{aligned}$$

Finally, the policy and value functions are given by

$$k'^*(k) = \alpha\beta Ak^\alpha$$

$$c^*(k) = (1 - \alpha\beta)Ak^\alpha$$

$$V(k) = \frac{1}{1 - \alpha\beta} \ln(Ak^\alpha) + \frac{\beta \ln A + \ln(\alpha\beta)^{\alpha\beta} + \ln(1 - \alpha\beta)^{(1 - \alpha\beta)}}{(1 - \beta)(1 - \alpha\beta)}$$

Notice that, had we solved the problem in terms of the state variable  $y = Ak^\alpha$ , the policy and value functions would be

$$k'^*(y) = \alpha\beta y$$

$$c^*(y) = (1 - \alpha\beta)y$$

$$V(y) = \frac{1}{1 - \alpha\beta} \ln y + \frac{\beta \ln A + \ln(\alpha\beta)^{\alpha\beta} + \ln(1 - \alpha\beta)^{(1 - \alpha\beta)}}{(1 - \beta)(1 - \alpha\beta)}$$

## 6. The McCall job search model

- ▶ The McCall search model :cite:'McCall1970' helped transform economists' way of thinking about labor markets.
- ▶ To clarify vague notions such as "involuntary" unemployment, McCall modeled the decision problem of unemployed agents directly, in terms of factors such as
  - ▶ current and likely future wages
  - ▶ impatience
  - ▶ unemployment compensation
- ▶ To solve the decision problem he used dynamic programming.
- ▶ Here we set up McCall's model and adopt the same solution method.
- ▶ As we'll see, McCall's model is not only interesting in its own right but also an excellent vehicle for learning dynamic programming.

# The McCall Model

- ▶ An unemployed worker receives in each period a job offer at wage  $W_t$ .
- ▶ At time  $t$ , our worker has two choices:
  1. **Accept the offer** and work permanently at constant wage  $W_t$ .
  2. **Reject the offer**, receive unemployment compensation  $c$ , and reconsider next period.
- ▶ The wage sequence is assumed to be IID with probability mass function  $\phi$ .
- ▶ Thus  $\phi(w)$  is the probability of observing wage offer  $w$  in the set  $\{w_1, \dots, w_n\}$ .

- ▶ The worker is infinitely lived and aims to maximize the expected discounted sum of earnings

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t Y_t$$

- ▶ The constant  $\beta$  lies in  $(0, 1)$  and is called a **discount factor**.
- ▶ The variable  $Y_t$  is income, equal to
  - ▶ his wage  $W_t$  when employed
  - ▶ unemployment compensation  $c$  when unemployed



# A Trade-Off

- ▶ The worker faces a trade-off:
  - ▶ Waiting too long for a good offer is costly, since the future is discounted.
  - ▶ Accepting too early is costly, since better offers might arrive in the future.
- ▶ To decide optimally in the face of this trade-off, we use dynamic programming.
- ▶ Dynamic programming can be thought of as a two-step procedure that
  1. first assigns values to "states" and
  2. then deduces optimal actions given those values
- ▶ We'll go through these steps in turn.

# The Value Function

- ▶ In order to optimally trade-off current and future rewards, we need to think about two things:
  1. the current payoffs we get from different choices
  2. the different states that those choices will lead to in next period (in this case, either employment or unemployment)
- ▶ To weigh these two aspects of the decision problem, we need to assign **values** to states.
- ▶ To this end, let  $V(w)$  be the total lifetime **value** accruing to an unemployed worker who enters the current period unemployed but with wage offer  $w$  in hand.
- ▶ More precisely,  $V(w)$  denotes the value of the objective function when an agent in this situation makes **optimal** decisions now and at all future points in time.

- ▶ Of course  $V(w)$  is not trivial to calculate because we don't yet know what decisions are optimal and what aren't!
- ▶ But think of  $V$  as a function that assigns to each possible wage  $w$  the maximal lifetime value that can be obtained with that offer in hand.
- ▶ A crucial observation is that this function  $V$  must satisfy the recursion

$$V(w) = \max \left\{ \frac{w}{1-\beta}, c + \beta \sum_{w'} V(w') \phi(w') \right\}$$

for every possible  $w$  in  $\{w_1, \dots, w_n\}$ .

- ▶ This important equation is a version of the **Bellman equation**, which is ubiquitous in economic dynamics and other fields involving planning over time.

$$V(w) = \max \left\{ \frac{w}{1-\beta}, c + \beta \sum_{w'} V(w') \phi(w') \right\}$$

- ▶ The intuition behind it is as follows:
  1. the first term inside the max operation is the lifetime payoff from accepting current offer  $w$ , since

$$w + \beta w + \beta^2 w + \dots = \frac{w}{1-\beta}$$

2. the second term inside the max operation is the **continuation value**, which is the lifetime payoff from rejecting the current offer and then behaving optimally in all subsequent periods
- ▶ If we optimize and pick the best of these two options, we obtain maximal lifetime value from today, given current offer  $w$ .
  - ▶ But this is precisely  $V(w)$ , which is the l.h.s. of the Bellman equation.

# The Optimal Policy

- ▶ Suppose for now that we are able to solve the Bellman equation for the unknown function  $V$ .
- ▶ Once we have this function in hand we can behave optimally (i.e., make the right choice between accept and reject).
- ▶ All we have to do is select the maximal choice on the r.h.s. of the Bellman equation.
- ▶ The optimal action is best thought of as a **policy**, which is, in general, a map from states to actions.
- ▶ In our case, the **state** is the current wage offer  $w$ .

- ▶ Given any  $w$ , we can read off the corresponding best choice (accept or reject) by picking the max on the r.h.s. of the Bellman equation.
- ▶ Thus, we have a map from  $\mathbb{R}$  to  $\{0, 1\}$ , with 1 meaning accept and 0 meaning reject.
- ▶ We can write the policy as follows

$$\sigma(w) := \mathbf{1} \left\{ \frac{w}{1 - \beta} \geq c + \beta \sum_{w'} V(w') \phi(w') \right\}$$

- ▶ Here  $\mathbf{1}\{P\} = 1$  if statement  $P$  is true and equals 0 otherwise.

# The Reservation Wage

- ▶ We can also write the policy function as

$$\sigma(w) := \mathbf{1}\{w \geq \bar{w}\}$$

where






$$\bar{w} := (1 - \beta) \left\{ c + \beta \sum_{w'} V(w') \phi(w') \right\}$$

- ▶ Here  $\bar{w}$  is a constant depending on  $\beta, c$  and the wage distribution called the **reservation wage**.
- ▶ The agent should accept if and only if the current wage offer exceeds the reservation wage.
- ▶ Clearly, we can compute this reservation wage if we can compute the value function.

# Computing the Optimal Policy

- ▶ Solving this model requires numerical methods, which are beyond the scope of this course.
- ▶ Those interested in this topic should take a look at [https://python.quantecon.org/mccall\\_model.html](https://python.quantecon.org/mccall_model.html)
- ▶ Indeed, the notes on this section on the McCall search model were taken from this website, which is part of the <https://quantecon.org/> website by Sargent and Stachurski.



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