Markov processes

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Table of contents

Introduction

- Markov processes are an indispensable ingredient of DSGE models.
- They preserve the recursive structure that these models inherit from their deterministic relatives.
- In this lecture we review a few results about these processes that we will need repeatedly in the modeling of business cycles.

1. Stochastic process

Stochastic Process

A stochastic process is a time sequence of random variables $\{Y_t\}_{t=-\infty}^{\infty}$.

Two types of processes:

Continuous if realizations are taken from an interval of the real line $Y_t \in [a, b] \subseteq \mathbb{R}$.

Discrete if there is a countable number of realizations $Y_t \in \{y_1, y_2, \ldots, y_n\}$.

i.i.d. Stochastic Process

- The elements of a stochastic process are identically and independently distributed (iid for short), if the probability distribution is the same for each member of the process Z_t and independent of the realizations of other members of the process.
- In this case

$$\mathbb{P}[Y_1 = y_1, Y_2 = y_2, \dots, Y_T = y_T] = \mathbb{P}(Y_1 = y_1) \times \mathbb{P}(Y_2 = y_2) \times \cdots \times \mathbb{P}(Y_T = y_T)$$

Unconditional moments

- Unconditional cumulative distribution function

$$F_{Y_t}(y) = \mathbb{P}\left[Y_t \leq y\right]$$

- Unconditional expectation (mean)

$$\mu_t \equiv \mathbb{E}\left(Y_t\right) = \int_{-\infty}^{\infty} y \, \mathrm{d}F_{Y_t}(y)$$

- Unconditional variance

$$\gamma_{0t} \equiv \mathbb{E}\left(Y_t - \mu_t\right)^2 = \int_{-\infty}^{\infty} \left(y - \mu_t\right)^2 \, \mathrm{d}F_{Y_t}(y)$$

- Autocovariance

$$\gamma_{jt} \equiv \mathbb{E}\left(Y_t - \mu_t\right)\left(Y_{t-j} - \mu_{t-j}\right)$$

Stationarity

If neither the mean μ_t nor the autocovariances γ_{jt} depend on the date t, then the process for Z_t is said to be covariance-stationary or weakly stationary:

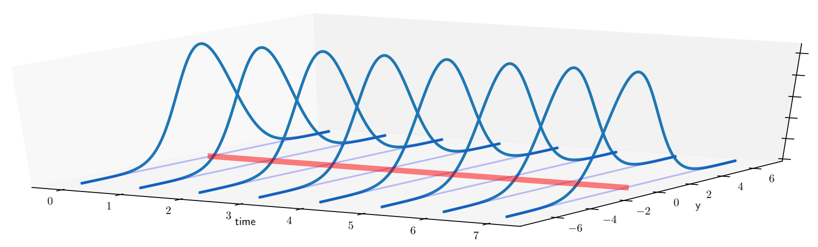
$$\begin{aligned} \mathbb{E}\left(Y_t\right) &= \mu && \text{for all } t \\ \mathbb{E}\left(Y_t - \mu\right)\left(Y_{t-j} - \mu\right) &= \gamma_j && \text{for all } t \text{ and any } j \end{aligned}$$

Example 1:

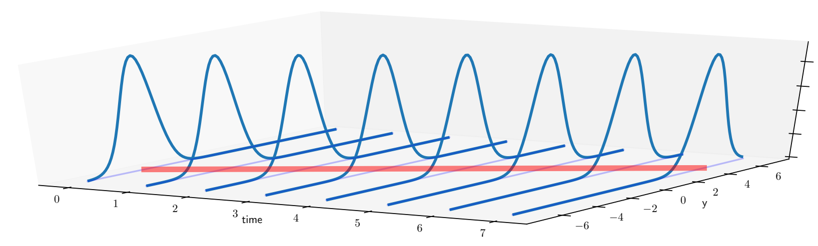
Stationary and nonstationary processes

Suppose Y_t is a stochastic process such that $Y_t \sim N(\mu_t, \sigma_t^2)$

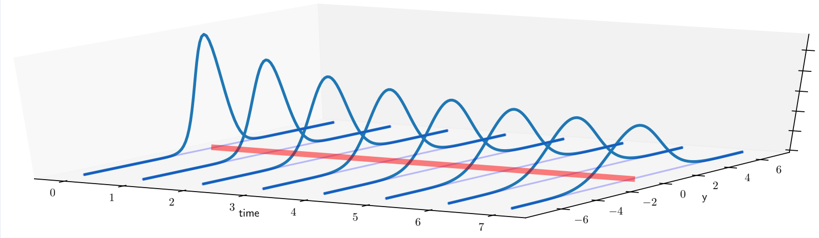
Stationary because μ_t and σ_t^2 are constant.



Nonstationary because μ_t is changing over time.



Nonstationary because σ_t^2 is changing over time.



White noise

- The basic building block for the processes considered in this lecture is a sequence $\{\epsilon_t\}$ whose elements have mean zero and variance σ^2,

$$\begin{aligned} \mathbb{E}\left(\epsilon_t\right) &= 0 && \text{(zero mean)} \\ \mathbb{E}\left(\epsilon_t^2\right) &= \sigma^2 && \text{(constant variance)} \\ \mathbb{E}\left(\epsilon_t \epsilon_\tau\right) &= 0 \quad \text{for } t \neq \tau && \text{(uncorrelated terms)} \end{aligned}$$

- If the terms are normally distributed

$$\epsilon_t \sim N(0, \sigma^2)$$

then we have the Gaussian white noise process.

2. The first-order autoregressive process

Definition of a AR(1) process

- A first-order autoregression, denoted AR(1), satisfies the following difference equation:

$$Y_t = c + \phi Y_{t-1} + \epsilon_t$$

  where $\{\epsilon_t\}$ is a white noise sequence.
- It is stationary if and only if $|\phi| < 1$.
- In what follows, we assume the process is stationary.

MA(∞) representation of a AR(1) process

- If the AR(1) process is stationary, it can be written

$$Y_t = \frac{c}{1-\phi} + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \phi^3\epsilon_{t-3} + \dots$$

Conditional versus unconditional mean

- The conditional mean given the previous observation is

$$\mathbb{E}[Y_t \mid Y_{t-1}] = c + \phi Y_{t-1}$$

- The unconditional mean is

$$\mu \equiv \mathbb{E}[Y_t] = \frac{c}{1-\phi}$$

- Since $c = (1-\phi)\mu$, the AR(1) process can be written as deviations from 'equilibrium'

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \epsilon_t$$

Impulse-response

- Starting with Y_{t-1}, the value of Y_{t+s} will be

$$Y_{t+s}-\mu = \phi^{s+1}(Y_{t-1}-\mu)+\phi^{s}\epsilon_t+\phi^{s-1}\epsilon_{t+1}+\cdots+\phi\epsilon_{t+s-1}+\epsilon_{t+s}$$

- Suppose that starting in 'equilibrium' ($Y_{t-1} - \mu = 0$) there is a time-t transitory shock ($\epsilon_t = \nu$) but no more shocks thereafter ($\epsilon_{t+1} = \cdots = \epsilon_{t+s} = 0$). Then

$$Y_{t+s} - \mu = \phi^{s}\nu$$

- This is known as an impulse-response function.
- Notice that the process will return to equilibrium as long as $|\phi| < 1$.

Conditional versus unconditional variance

- The conditional variance given the previous observation is

$$\text{Var}[Y_t \mid Y_{t-1}] = \text{Var}[c + \phi Y_{t-1} + \epsilon_t \mid Y_{t-1}] = \sigma^2$$

- The unconditional mean is

$$\gamma_0 \equiv \text{Var}[Y_t] = \frac{\sigma^2}{1-\phi^2}$$

- Notice that $\gamma_0 > \text{Var}[Y_t \mid Y_{t-1}]$

Autocovariance and autocorrelation

- The autocovariance is given by

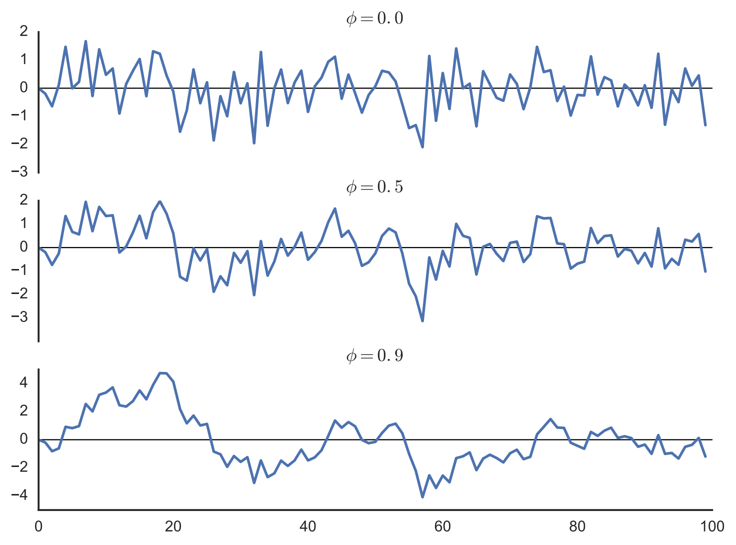
$$\gamma_j = \phi^j \gamma_0 \qquad (j = 1, 2, \dots)$$

- The autocorrelation is given by

$$\rho_j = \phi^j \qquad (j = 1, 2, \dots)$$

Example 2:

Realizations of an AR(1) process



The three processes are build from the same white noise realization. Notice how process becomes more persistent as ϕ approaches 1.

3. Markov chains

Markov property

A stochastic process $\{Z_t\}_{t=0}^{\infty}$ has the Markov property if for all $k \geq 1$ and all t,

$$\mathbb{P}[Z_{t+1} \,|\, Z_t, Z_{t-1}, \ldots, Z_{t-k}] = \mathbb{P}[Z_{t+1} \,|\, Z_t]$$

That is, the the probability distribution of Z_{t+1} only depends upon the realization of Z_t.

Example 3:

AR(1) process

- The AR(1) process is a Markov process:

$$Z_{t+1} = (1-\rho)\bar{Z} + \rho Z_t + \epsilon_{t+1}$$

where $\rho \in [0, 1)$, and $\epsilon_{t+1} \sim \text{iid} N(0, \sigma^2)$ is a white noise process.

- Given Z_t, next period's variable Z_{t+1} is normally distributed with:

$$\begin{aligned} \text{mean:} \quad & \mathbb{E}(Z_{t+1} \mid Z_t) = (1-\rho)\bar{Z} + \rho Z_t \\ \text{variance:} \quad & \operatorname{Var}(Z_{t+1} \mid Z_t) = \sigma^2 \end{aligned}$$

Markov Chains

Markov chains are discrete valued Markov processes. They are characterized by three objects:

1. The n different realizations of Z_t, represented by the column vector $z = [z_1, z_2, \dots, z_n]'$.
2. The probability distribution of the initial date $t = 0$, $\pi_0 = [\pi_{01}, \pi_{02}, \dots, \pi_{0n}]'$, where $\pi_{0i} = \mathbb{P}[Z_0 = z_i]$.
3. The transition matrix $P = (p_{ij})$, where $p_{ij} = \mathbb{P}[Z_{t+1} = z_j \mid Z_t = z_i]$, representing the dynamics of the process.

Notice that

- $p_{ij} \geq 0$ and $\sum_{j=1}^{n} p_{ij} = 1$.
- $\pi_{0i} \geq 0$ and $\sum_{i=1}^{n} \pi_{0i} = 1$.

Example 4:
Unemployment

A worker can either be employed or unemployed:

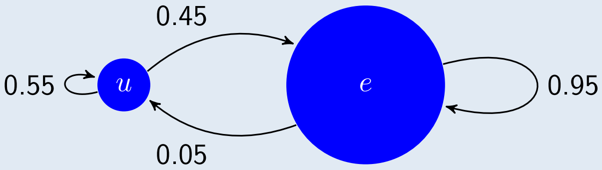
- If unemployed, she will get a job with probability $p = 45\%$
- If employed, she will lose her job with probability $q = 5\%$

The worker is employed at $t = 0$. Then the Markov chain is:

outcomes $\{$unemployed, employed$\}$ or $z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

initial probability $\pi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

transition probability $P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \\ 0.05 & 0.95 \end{bmatrix}$



Example 5:
Credit ratings

Transition of the credit ratings from one year to the next:

	AAA	AA	A	BBB	BB	B	CCC	D	N.R.
AAA	90.34	5.62	0.39	0.08	0.03	0	0	0	3.5
AA	0.64	88.78	6.72	0.47	0.06	0.09	0.02	0.01	3.21
A	0.07	2.16	87.94	4.97	0.47	0.19	0.01	0.04	4.16
BBB	0.03	0.24	4.56	84.26	4.19	0.76	0.15	0.22	5.59
BB	0.03	0.06	0.4	6.09	76.09	6.82	0.96	0.98	8.58
B	0	0.09	0.29	0.41	5.11	74.62	3.43	5.3	10.76
CCC	0.13	0	0.26	0.77	1.66	8.93	53.19	21.94	13.14
D	0	0	0	0	1	3.1	9.29	51.29	37.32
N.R.	0	0	0	0	0	0.1	8.55	74.06	17.07

Transition probabilities are expressed in %.

- Higher ratings are more stable: the diagonal coefficients of the matrix go decreasing.
- Starting from the rating AA it is easier to be downgraded (probability 6.72%) than to be upgraded (probability 0.64%).

This figure shows a simulation of a bond rating, assuming that it starts as a AAA bond.

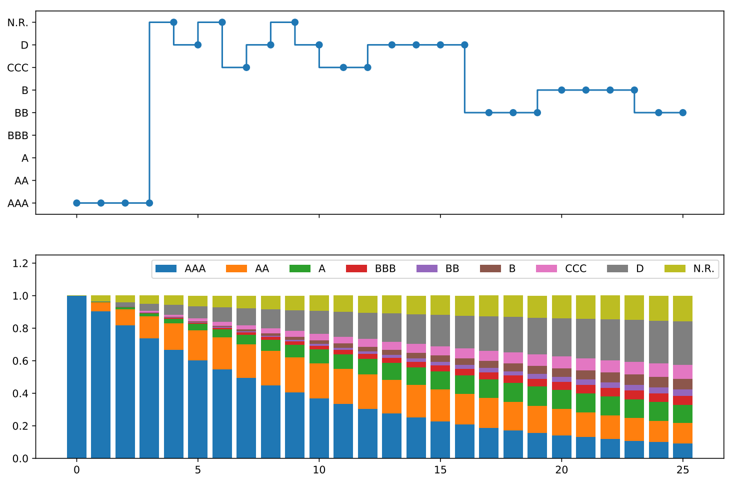


Figure below shows the evolution of the unconditional distribution (to be studied later).

Transition over multiple periods

- The transition matrix is also called a stochastic matrix.
- It defines the probabilities of moving from one value of the state to another in one period.
- The probability of moving from one value of the state to another in two periods is determined by P2 because

$$
\begin{aligned}
&\mathbb{P}[Z_{t+2} = z_j | Z_t = z_i] \\
&\quad = \sum_{h=1}^{n} \mathbb{P}[Z_{t+2} = z_j | Z_{t+1} = z_h] \times \mathbb{P}[Z_{t+1} = z_h | Z_t = z_i] \\
&\quad = \sum_{h=1}^{n} P_{ih} P_{hj} = P_{ij}^{(2)}
\end{aligned}
$$

The unconditional distribution

The probability distribution of Z_t evolves according to $\pi'_{t+1} = \pi'_t P$. Therefore

$$\begin{aligned}
\pi'_1 &= \pi'_0 P \\
\pi'_2 &= \pi'_0 P^2 \\
&\vdots \\
\pi'_k &= \pi'_0 P^k
\end{aligned}$$

The limit for $k \to \infty$ is the *time invariant, stationary, or ergodic* distribution of the Markov chain. It is defined by

$$\pi' = \pi' P \quad \Leftrightarrow \quad (I - P')\pi = 0$$

The limit exist and is independent of the initial distribution π_0 if $p_{ij}^{(k)} > 0$ for some integer $k \geq 1$.

Example 6:
Unemployment (cont.)

For the worker who can either be employed or unemployed according to Markov matrix

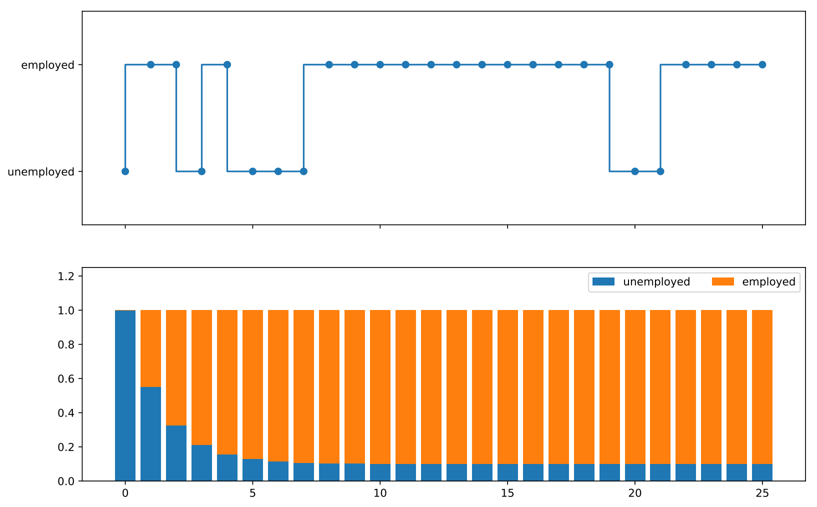
$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \\ 0.05 & 0.95 \end{bmatrix}$$

the stationary distribution $\begin{bmatrix} x & 1-x \end{bmatrix}'$ is the solution to:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix} \right\} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} p & -q \\ -p & q \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then $x = \frac{q}{p+q}$ and the stationary distribution is: $\begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}$. This means that the long run probability of being unemployed is 10%.

This figure shows a simulation of the employment status, assuming that $\pi_0 = [1, 0]'$ (that is, the worker is unemployed in period $t = 0$)



Example 7:

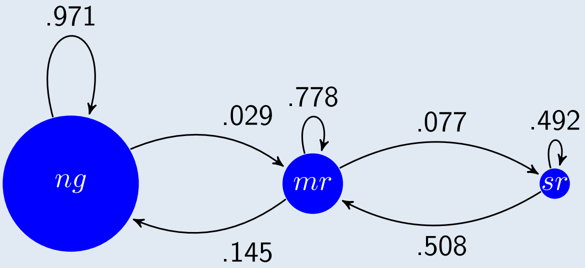
Business Cycle

Using monthly data on US unemployment, Hamilton estimated this stochastic matrix

$$P = \begin{bmatrix} 0.971 & 0.029 & 0.000 \\ 0.145 & 0.778 & 0.077 \\ 0.000 & 0.508 & 0.492 \end{bmatrix}$$

where the states are { "normal growth", "mild recession", "severe recession"}

The transition matrix can also be represented by:



To find the stationary distribution:

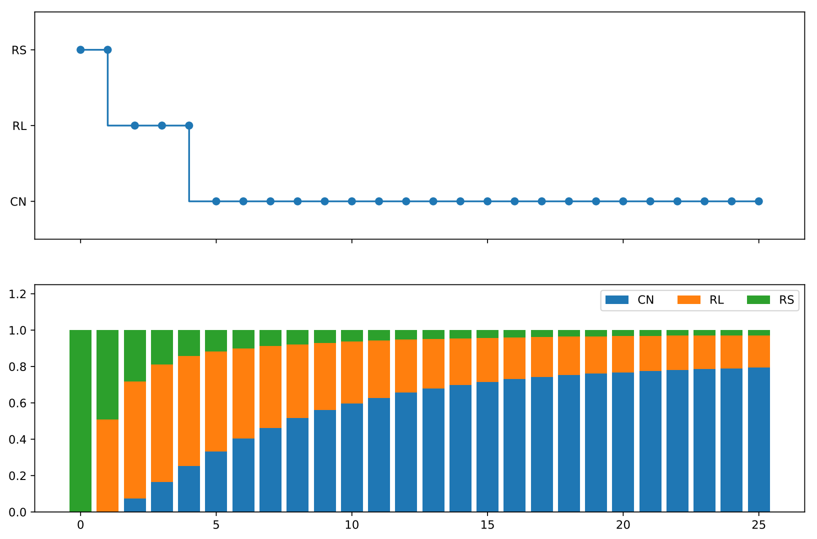
$$\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0.971 & 0.145 & 0.000\\ 0.029 & 0.778 & 0.508\\ 0.000 & 0.077 & 0.492\end{bmatrix}\right)\begin{bmatrix}x\\ y\\ 1-x-y\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0.029 & -0.145 & 0.000\\ -0.029 & 0.222 & -0.508\\ 0.000 & -0.077 & 0.508\end{bmatrix}\begin{bmatrix}x\\ y\\ 1-x-y\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

- We need only two of the equations (system is linearly dependent).
- From first equation, we conclude that $0.029x = 0.145y \Rightarrow x = 5y$
- From last one, $-0.077y + 0.508(1-6y) = 0 \Rightarrow y = 0.16256$
- Thus, the stationary distribution is:

$$\pi = \begin{bmatrix}0.81280\\ 0.16256\\ 0.02464\end{bmatrix}$$

This figure shows a simulation of the business cycle, assuming that $\pi_0 = \left[0, 0, 1\right]'$ (economy starts in a severe recession).



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