Universidad de Costa Rica
EC3201 - Teoría Macroeconómica 2

Practice 2: The CES utility function

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A consumer problem: 2 goods

A consumer gets utility from consuming goods x and y according to a utility function:

$$U(x,y) \tag{1}$$

The prices of the goods are p_x and p_y, and the consumer has available (nominal) income M. The budget constraint is therefore

$$p_x x + p_y y = M \tag{2}$$

The consumer wants to get as much utility as possible, given the market prices and his income. The Lagrangean for this optimization problem is

$$\mathcal{L}(x,y,\lambda) = U(x,y) + \lambda(M - p_x x - p_y y) \tag{3}$$

where λ is the Lagrange multiplier associated with the budget constraint. In the optimal allocation $(x^*,\ y^*)$, it must be the case that

$$U_x\left(x^*,\ y^*\right) = \lambda p_x \tag{4a}$$
$$U_y\left(x^*,\ y^*\right) = \lambda p_y \tag{4b}$$
$$p_x x^* + p_y y^* = M \tag{4c}$$

where U_x and U_y denote the partial derivatives of the utility function with respect to x and to y, respectively.

A CES utility function

In what follows, we assume that the utility function takes the form of a CES function:

$$U(x,y) = \left(\theta x^\rho + (1-\theta) y^\rho\right)^{1/\rho} \tag{5}$$

and we define the price index P by

$$P\left(p_x, p_y\right) \equiv \left[\theta^\sigma p_x^{1-\sigma} + (1-\theta)^\sigma p_y^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{6}$$

where $\rho < 1$ and $\sigma \equiv \frac{1}{1-\rho}$.

1. (a) Prove that both the utility function and the price index are homogeneous of degree one.

Solution: Let $k > 0$ be a constant. For the utility function:

$$\begin{aligned} U(kx, ky) &= \left[\theta(kx)^\rho + (1-\theta)(ky)^\rho\right]^{1/\rho} \\ &= \left[\theta k^\rho x^\rho + (1-\theta)k^\rho y^\rho\right]^{1/\rho} \\ &= \left[k^\rho\left(\theta x^\rho + (1-\theta)y^\rho\right)\right]^{1/\rho} \\ &= k^{\rho/\rho}\left(\theta x^\rho + (1-\theta)y^\rho\right)^{1/\rho} \\ &= kU(x, y) \end{aligned}$$

Similarly, for the price index

$$\begin{aligned} P(kp_x, kp_y) &= \left[\theta^\sigma (kp_x)^{1-\sigma} + (1-\theta)^\sigma (kp_y)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \\ &= \left[\theta^\sigma k^{1-\sigma} p_x^{1-\sigma} + (1-\theta)^\sigma k^{1-\sigma} p_y^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \\ &= \left[k^{1-\sigma}\left(\theta^\sigma p_x^{1-\sigma} + (1-\theta)^\sigma p_y^{1-\sigma}\right)\right]^{\frac{1}{1-\sigma}} \\ &= k^{\frac{1-\sigma}{1-\sigma}}\left(\theta^\sigma p_x^{1-\sigma} + (1-\theta)^\sigma p_y^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \\ &= kP(p_x, p_y) \end{aligned}$$

(b) Show that the marginal utilities with respect to the goods are given by

$$U_x \equiv \frac{\partial U}{\partial x} = \theta\left(\frac{U}{x}\right)^{1-\rho} \qquad \text{and} \qquad U_y \equiv \frac{\partial U}{\partial y} = (1-\theta)\left(\frac{U}{y}\right)^{1-\rho}$$

[Hint: raise both sides of (5) to the power of ρ]

Solution: From (5):

$$U^\rho = \theta x^\rho + (1-\theta)y^\rho$$

Taking implicit derivative wrt x:

$$\rho U^{\rho-1}\frac{\partial U}{\partial x} = \rho\theta x^{\rho-1}$$

Solving for $U_x = \frac{\partial U}{\partial x}$:

$$U_x = \theta\left(\frac{U}{x}\right)^{1-\rho}$$

The same procedure is used to show that

$$U_y = (1-\theta)\left(\frac{U}{y}\right)^{1-\rho}$$

(c) Dividing (4a) by (4b), we find that in the optimal allocation it must be the case that

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

Use this result to show that the optimal allocation x^* and y^* must satisfy the following condition:

$$\frac{x^*}{y^*} = \left(\frac{\theta p_y}{(1-\theta)p_x}\right)^\sigma \tag{7}$$

Solution: Substituting the marginal utilities in the optimality condition we get

$$\begin{aligned}
\frac{U_x}{U_y} &= \frac{p_x}{p_y} \\
\frac{\theta\left(\frac{U}{x}\right)^{1-\rho}}{(1-\theta)\left(\frac{U}{y}\right)^{1-\rho}} &= \frac{p_x}{p_y} \\
\frac{x^{\rho-1}}{y^{\rho-1}} &= \frac{(1-\theta)p_x}{\theta p_y} \\
\frac{x}{y} &= \left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\frac{1}{1-\rho}}
\end{aligned}$$

(d) Use (2) and (7) to prove that the demand for goods are given by:

$$x^*(p_x, p_y, M) = \left(\frac{\theta}{p_x/P}\right)^{\sigma}\frac{M}{P} \qquad\qquad y^*(p_x, p_y, M) = \left(\frac{1-\theta}{p_y/P}\right)^{\sigma}\frac{M}{P} \tag{8}$$

Solution: From (7) we have $x^* = \left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\sigma} y^*$. Substitute into the budget constraint

$$\begin{aligned}
M &= p_x x^* + p_y y^* \\
&= p_x\left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\sigma} y^* + p_y y^* \\
&= \left[p_x\left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\sigma} + p_y\right] y^* \\
&= \left[\frac{\theta^{\sigma}}{(1-\theta)^{\sigma}} p_x^{1-\sigma} p_y^{\sigma} + p_y\right] y^* \\
&= \left[\frac{\theta^{\sigma} p_x^{1-\sigma} p_y^{\sigma} + (1-\theta)^{\sigma} p_y}{(1-\theta)^{\sigma}}\right] y^* \\
&= \left[\theta^{\sigma} p_x^{1-\sigma} + (1-\theta)^{\sigma} p_y^{1-\sigma}\right]\left[\frac{p_y^{\sigma}}{(1-\theta)^{\sigma}}\right] y^* \\
&= P^{1-\sigma}\left(\frac{p_y}{1-\theta}\right)^{\sigma} y^* \\
&= P\left(\frac{p_y/P}{1-\theta}\right)^{\sigma} y^* \\
\Rightarrow y^* &= \left(\frac{1-\theta}{p_y/P}\right)^{\sigma}\frac{M}{P}
\end{aligned}$$

To find x^* we substitute for y^* using the first equation in this solution:

$$\begin{aligned}
x^* &= \left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\sigma} y^* \\
&= \left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\sigma}\left(\frac{1-\theta}{p_y/P}\right)^{\sigma}\frac{M}{P} \\
&= \left(\frac{\theta}{p_x/P}\right)^{\sigma}\frac{M}{P}
\end{aligned}$$

(e) Show that the Lagrange multiplier associated with the budget constraint equals the inverse of the price index, that is: $\lambda^* = P^{-1}$. To do so, you can follow these steps:

1. Knowing that U is homogeneous of degree one, find an expression for $\frac{U}{y}$. The result should depend on the ratio $\frac{x}{y}$
2. Use the result of step 1 to compute the marginal utility of y (see question (b)).
3. Use step 2 and equation (4b) to obtain an expression for λ that depends on $\frac{x}{y}$.
4. Replace $\frac{x}{y}$ from step 3 with equation (7).
5. Finally, simplify the result, keeping in mind that $\sigma \equiv \frac{1}{1-\rho}$.

Solution: Using the homogeneity of U we get:

$$\begin{aligned}\frac{U}{y} &= \tfrac{1}{y}U = \tfrac{1}{y}\left(\theta x^\rho + (1-\theta)y^\rho\right)^{1/\rho} \\ &= \left[\theta\left(\frac{x}{y}\right)^\rho + (1-\theta)\right]^{1/\rho}\end{aligned}$$

Then, the marginal utility of y is

$$\begin{aligned}U_y &= (1-\theta)\left(\frac{U}{y}\right)^{1-\rho} \\ &= (1-\theta)\left[\theta\left(\frac{x}{y}\right)^\rho + (1-\theta)\right]^{\frac{1-\rho}{\rho}}\end{aligned}$$

Using the first-order condition (4b):

$$\begin{aligned}\lambda^* &= \frac{U_y}{p_y} \\ &= \frac{1-\theta}{p_y}\left[\theta\left(\frac{x}{y}\right)^\rho + (1-\theta)\right]^{\frac{1-\rho}{\rho}} \\ &= \left\{\left(\frac{1-\theta}{p_y}\right)^{\frac{\rho}{1-\rho}}\left[\theta\left(\frac{x}{y}\right)^\rho + (1-\theta)\right]\right\}^{\frac{1-\rho}{\rho}} \\ &= \left\{\left(\frac{1-\theta}{p_y}\right)^{\frac{\rho}{1-\rho}}\left[\theta\left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\sigma\rho} + (1-\theta)\right]\right\}^{\frac{1-\rho}{\rho}}\end{aligned}$$

where the last step follows from using (7). Because $\sigma \equiv \frac{1}{1-\rho}$, it is easy to show that $\sigma\rho = \frac{\rho}{1-\rho} = \sigma - 1$. Then, in last expression

$$\begin{aligned}\lambda^* &= \left\{\left(\frac{1-\theta}{p_y}\right)^{\sigma-1}\left[\theta\left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\sigma-1} + (1-\theta)\right]\right\}^{\frac{1}{\sigma-1}} \\ &= \left[\theta\left(\frac{1-\theta}{p_y}\right)^{\sigma-1}\left(\frac{\theta p_y}{(1-\theta)p_x}\right)^{\sigma-1} + (1-\theta)\left(\frac{1-\theta}{p_y}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} \\ &= \left[\theta\left(\frac{\theta}{p_x}\right)^{\sigma-1} + (1-\theta)\left(\frac{1-\theta}{p_y}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} \\ &= \left\{\left[\theta^\sigma p_x^{1-\sigma} + (1-\theta)^\sigma p_y^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\right\}^{-1} \\ &= P^{-1}\end{aligned}$$

(f) Use the previous result to show that the partial derivative of the Lagrangean with respect to the nominal income M equals P^{-1}.

Solution: Using (3), the partial derivative of $\mathcal{L}$ with respect to M equals λ, which in the previous exercise we already showed equals P^-1

(g) Substitute (8) into (5) to prove that the *indirect utility function*[1] is given by:

$$V(p_x, p_y, M) \equiv U(x^*, y^*) = \frac{M}{P} \tag{9}$$

Solution:

$$\begin{aligned}
U(x^*, y^*) &= (\theta x^{*\rho} + (1-\theta) y^{*\rho})^{1/\rho} \\
&= \left[\theta \left(\frac{\theta}{p_x/P}\right)^{\sigma\rho} \left(\frac{M}{P}\right)^{\rho} + (1-\theta)\left(\frac{1-\theta}{p_y/P}\right)^{\sigma\rho}\left(\frac{M}{P}\right)^{\rho}\right]^{1/\rho} \\
&= \left(\frac{M}{P}\right)\left[\theta \left(\frac{\theta}{p_x/P}\right)^{\sigma-1} + (1-\theta)\left(\frac{1-\theta}{p_y/P}\right)^{\sigma-1}\right]^{1/\rho} \qquad \text{(notice that } \sigma\rho = \sigma - 1\text{)} \\
&= \left(\frac{M}{P}\right)\left[\theta^{\sigma} p_x^{1-\sigma} P^{\sigma-1} + (1-\theta)^{\sigma} p_y^{1-\sigma} P^{\sigma-1}\right]^{1/\rho} \\
&= \left(\frac{M}{P}\right)\left[\frac{\theta^{\sigma} p_x^{1-\sigma} + (1-\theta)^{\sigma} p_y^{1-\sigma}}{P^{1-\sigma}}\right]^{1/\rho} \\
&= \frac{M}{P}
\end{aligned}$$

where the last step follows from the definition of the price index P.

(h) Compute the derivative of the indirect utility function with respect to M. Compare your result to that of question f.

Solution: It is straightforward to see that

$$\frac{\partial V(p_x, p_y, M)}{\partial M} = \frac{\partial \frac{M}{P}}{\partial M} = \frac{1}{P}$$

This is the same as $\frac{\partial \mathcal{L}}{\partial M}$ from question f, which we evaluated *as if* the optimal quantities x and y did not depend on income (which we know they do, from question d). This is an instance of the *envelope condition.*

(i) Using (7), show that elasticity of substitution of the goods is given by σ. That is, prove that

$$\frac{\Delta\% \left(\frac{x^*}{y^*}\right)}{\Delta\% \left(\frac{p_x}{p_y}\right)} = -\sigma$$

Solution: Taking logs in both sides of (7)

$$\begin{aligned}
\ln \frac{x^*}{y^*} &= \ln \left(\frac{\theta p_y}{(1-\theta) p_x}\right)^{\sigma} && (10) \\
&= \sigma \ln \theta - \sigma \ln(1-\theta) - \sigma \ln \frac{p_x}{p_y} && (11)
\end{aligned}$$

[1] The indirect utility function is a particular case of a *value function.*

Take derivatives of the ratio $\frac{x^*}{y^*}$ with respect to the relative price $\frac{p_x}{p_y}$

$$\frac{1}{\frac{x^*}{y^*}} \frac{\mathrm{d}\frac{x^*}{y^*}}{\mathrm{d}\frac{p_x}{p_y}} = \frac{-\sigma}{\frac{p_x}{p_y}} \tag{12}$$

Therefore

$$\frac{\frac{\mathrm{d}\frac{x^*}{y^*}}{\frac{x^*}{y^*}}}{\frac{\mathrm{d}\frac{p_x}{p_y}}{\frac{p_x}{p_y}}} = \frac{\Delta\%\left(\frac{x^*}{y^*}\right)}{\Delta\%\left(\frac{p_x}{p_y}\right)} = -\sigma$$

(j) Optional: show that

$$\lim_{\rho\to 0} U(x,y) = x^\theta y^{1-\theta}$$

that is, that the Cobb-Douglas is a special case of the CES function where $\rho = 0$. Hint: Remember that $\lim_{z\to 0} g(z) = e^{\lim_{z\to 0}\ln g(z)}$. Hint 2: You will need L'Hôpital rule.

Solution:

$$\begin{aligned}
\lim_{\rho\to 0} U(x,y) &= \lim_{\rho\to 0} \left(\theta x^\rho + (1-\theta)y^\rho\right)^{1/\rho} \\
&= \exp\left[\lim_{\rho\to 0} \frac{\ln\left(\theta x^\rho + (1-\theta)y^\rho\right)}{\rho}\right] \\
&= \exp\left[\lim_{\rho\to 0} \frac{\theta x^\rho \ln x + (1-\theta)y^\rho \ln y}{\theta x^\rho + (1-\theta)y^\rho}\right] \\
&= \exp\left[\frac{\theta \ln x + (1-\theta)\ln y}{\theta + (1-\theta)}\right] \\
&= \exp\left[\frac{\ln x^\theta + \ln y^{1-\theta}}{1}\right] \\
&= x^\theta y^{1-\theta}
\end{aligned}$$