Perturbation methods: Solving DSGE models with Dynare

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About this presentation

- Dynare is a user-friendly software platform useful for solving dynamic economic models.
- Dynare relies on perturbation methods to find solution to models.
- The two objectives of this presentation are:
    - to explain the logic behind the use of perturbation methods; and
    - to provide a short introduction to the use of Dynare.
- For illustration, we solve the Solow model and the Ramsey model.

Outline

Perturbation methods

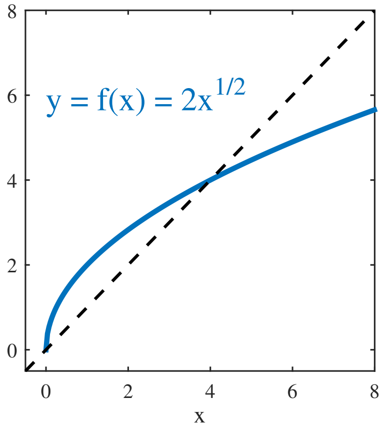
- Dynare approach to solving dynamic models is based on perturbation methods.
- These methods are based on Taylor's expansions and the implicit function theorem.
- In this section we review the mathematics necessary to understand perturbation methods.

Fixed point of a function

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function. A point $\mathbf{x}^* \in \mathbb{R}^n$ is called a *fixed point* of f if it satisfies

$$f(\mathbf{x}^*) = \mathbf{x}^*$$

Example: $f(x) = 2\sqrt{x}$ has two fixed points: $x^* = 0$ and $x^* = 4$.



Some definitions

Let f be a function, $f : \mathbb{R}^n \to \mathbb{R}$, where $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}'$. We denote the first partial derivatives of $f(\mathbf{x})$ by

$$f_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i} \quad \text{and} \quad \nabla f(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

and the Hessian matrix of $f(\mathbf{x})$ by

$$H(\mathbf{x}) = \begin{bmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \dots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \dots & f_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \dots & f_{nn}(\mathbf{x}) \end{bmatrix}$$

Some notation

Let f be a function, $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$f(\mathbf{x}) = \begin{pmatrix} f^1(\mathbf{x}) \\ \vdots \\ f^m(\mathbf{x}) \end{pmatrix}$$

We denote the Jacobian of $f(\mathbf{x})$ by

$$J(\mathbf{x}) = \begin{bmatrix} f_1^1(\mathbf{x}) & f_2^1(\mathbf{x}) & \dots & f_n^1(\mathbf{x}) \\ f_1^2(\mathbf{x}) & f_2^2(\mathbf{x}) & \dots & f_n^2(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^m(\mathbf{x}) & f_2^m(\mathbf{x}) & \dots & f_n^m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \nabla f^1(\mathbf{x})' \\ \nabla f^2(\mathbf{x})' \\ \vdots \\ \nabla f^m(\mathbf{x})' \end{bmatrix}$$

A partial Jacobian

- Let $g(\mathbf{x},\mathbf{y})$ be a function of vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, such that $g : \mathbb{R}^{n+m} \to \mathbb{R}^m$.
- Think of g as a system of m nonlinear equations on m endogenous variables $\mathbf{y}$ and n exogenous variables $\mathbf{x}$.
- The partial Jacobians Dg_x and Dg_y form a partition of the Jacobian:

$$J(\mathbf{x},\mathbf{y}) = \left[Dg_x \mid Dg_y\right] = \begin{bmatrix} g^1_{x_1} & g^1_{x_2} & \cdots & g^1_{x_n} & g^1_{y_1} & g^1_{y_2} & \cdots & g^1_{y_m} \\ g^2_{x_1} & g^2_{x_2} & \cdots & g^2_{x_n} & g^2_{y_1} & g^2_{y_2} & \cdots & g^2_{y_m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g^m_{x_1} & g^m_{x_2} & \cdots & g^m_{x_n} & g^m_{y_1} & g^m_{y_2} & \cdots & g^m_{y_m} \end{bmatrix}$$

Taylor's Theorem

Taylor's theorem, $\mathbb{R}$ case

Let $f : [a, b] \to \mathbb{R}$ be a $n+1$ times continuously differentiable function on (a,b), let $\bar{x}$ be a point in (a,b). Then

$$f(\bar{x}+h) = f(\bar{x}) + f^{(1)}(\bar{x})h + f^{(2)}(\bar{x})\frac{h^2}{2} + \ldots + f^{(n)}(\bar{x})\frac{h^n}{n!} + f^{(n+1)}(\xi)\frac{h^{n+1}}{(n+1)!}, \qquad \xi \in (\bar{x}, \bar{x}+h)$$

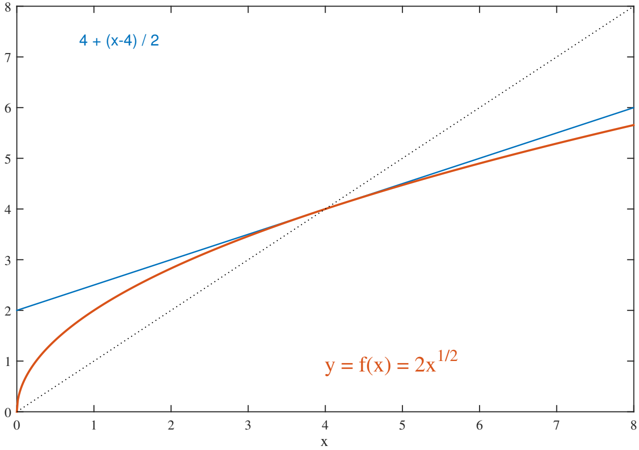
Example of Taylor's Theorem: $f(x) = 2\sqrt{x}$

Approximation of

$$f(x) = 2\sqrt{x}$$

around the fixed point

$$x^* = 4$$



Taylor approximations

Taylor's approximation, $\mathbb{R}^n$ case

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function on a neighborhood of $\mathbf{x}$. Then

$$f(\mathbf{x}+\mathbf{h}) \approx f(\mathbf{x}) + [\nabla f(\mathbf{x})]' \, \mathbf{h} + \tfrac{1}{2}\mathbf{h}' H(\mathbf{x})\mathbf{h}$$

is a second-order Taylor approximation of f at the point x

Taylor approximation around a fixed point

Suppose that the sequence $\mathbf{x}_{t+1} = f(\mathbf{x}_t)$ of vectors in $\mathbb{R}^n$ has a fixed point $\mathbf{x}^*$. Then the second-order Taylor approximation around $\mathbf{x}^*$ is

$$\mathbf{x}_{t+1} - \mathbf{x}^* \approx [\nabla f(\mathbf{x}^*)]' (\mathbf{x}_t - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x}_t - \mathbf{x}^*)' H(\mathbf{x})(\mathbf{x}_t - \mathbf{x}^*)$$

The first-order approximation is simply

$$\mathbf{x}_{t+1} - \mathbf{x}^* \approx [\nabla f(\mathbf{x}^*)]' (\mathbf{x}_t - \mathbf{x}^*)$$

These expressions can be interpreted as describing "deviations from equilibrium". Notice the similarity of the latter expression to a VAR(1) model written as deviation from the mean (without the stochastic term).

Implicit Function Theorem

Suppose $y = f(x)$ is a function of x, $f : \mathbb{R} \to \mathbb{R}$, but y is implicitly defined by:

$$0 = g(x, y) = g(x, f(x))$$

How to compute the derivative of y with respect to x at a point a? Simply take the derivative of g with respect to x:

$$0 = \frac{\partial g(x, y)}{\partial x} + \frac{\partial g(x, y)}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}x}$$

As long as $g_y \neq 0$, the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ at point a is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[\frac{\partial g(a, f(a))}{\partial y}\right]^{-1} \frac{\partial g(a, f(a))}{\partial x}$$

Implicit Function Theorem

Second derivative

We can use $0 = g_x(x, y) + g_y(x, y)y'$ to obtain the second derivative y'':

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} g_x(x, y) + y' \frac{\partial}{\partial x} g_y(x, y) + g_y(x, y) \frac{\partial}{\partial x} y' \\ &= g_{xx} + g_{xy} y' + y' \left(g_{yx} + g_{yy} y' \right) + g_y y'' \end{aligned}$$

Then, the second derivative is

$$y'' = -\frac{1}{g_y} \left[g_{xx} + 2 g_{xy} y' + g_{yy} (y')^2 \right]$$

Implicit Function Theorem, several variables

Now suppose that $\mathbf{y} = f(\mathbf{x})$ is a function, $f : \mathbb{R}^n \to \mathbb{R}^m$, but f is implicitly defined by m possibly nonlinear functions $g : \mathbb{R}^{n+m} \to \mathbb{R}^m$:

$$0 = g(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} g^1(\mathbf{x}, \mathbf{y}) \\ \vdots \\ g^m(\mathbf{x}, \mathbf{y}) \end{bmatrix}$$

The Jacobian of $f(x)$ at a point $\bar{\mathbf{x}}$ is given by

$$J(\bar{\mathbf{x}}) = -\left[Dg_y(\bar{\mathbf{x}}, f(\bar{\mathbf{x}}))\right]^{-1} Dg_x(\bar{\mathbf{x}}, f(\bar{\mathbf{x}}))$$

where the $m \times m$ matrix Dg_y is invertible.

Regular perturbation: the basic idea

- Suppose that a problem reduces to solving

$$f(x, \epsilon) = 0$$

  for x where ϵ is a parameter.
- We assume that for each value of ϵ the equation has a solution for x.
- Let $x = x(\epsilon)$ denote a smooth function such that $f(x(\epsilon), \epsilon) = 0$.
- In general, we cannot solve the equation for arbitrary ϵ, but there may be special values of ϵ for which the equation can be solved.

Example

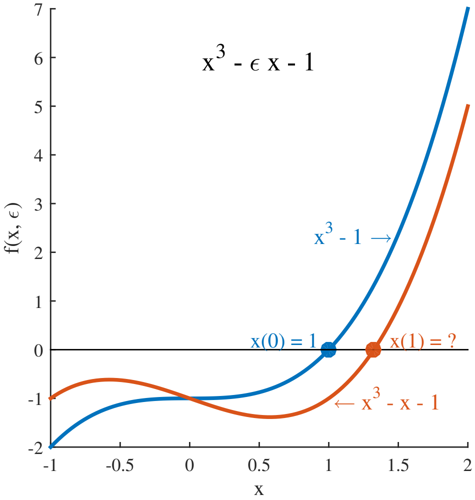
- Consider the function

$$f(x(\epsilon), \epsilon) = x^3 - \epsilon x - 1 = 0$$

- For $\epsilon = 0$, it's easy to find a solution:

$$f(x(0), 0) = x^3 - 1 = 0 \Rightarrow x(0) = 1$$

- We will approximate solutions to $x^3(\epsilon) - \epsilon x(\epsilon) - 1 = 0$ for arbitrary ϵ, based on knowing $x(0) = 1$.



Regular perturbation: the basic idea (cont.)

- If $f(x, \epsilon)$ and $x(\epsilon)$ are differentiable, and $x(0)$ is known, we can differentiate $f(x(\epsilon), \epsilon) = 0$ to get an implicit expression for $x'(\epsilon)$:

$$f_x(x(\epsilon), \epsilon)x'(\epsilon) + f_\epsilon(x(\epsilon), \epsilon) = 0$$

- Evaluated at *known* $x(0)$, it becomes

$$f_x(x(0), 0)x'(0) + f_\epsilon(x(0), 0) = 0 \Rightarrow x'(0) = -\frac{f_\epsilon(x(0),0)}{f_x(x(0),0)}$$

- Then, the linear approximation of $x(\epsilon)$ for ϵ near zero is

$$x(\epsilon) \approx x^L(\epsilon) = x(0) - \frac{f_\epsilon(x(0),0)}{f_x(x(0),0)}\,\epsilon$$

Example (cont.)

- For $f(x(\epsilon),\epsilon) = x^3 - \epsilon x - 1 = 0$ we get

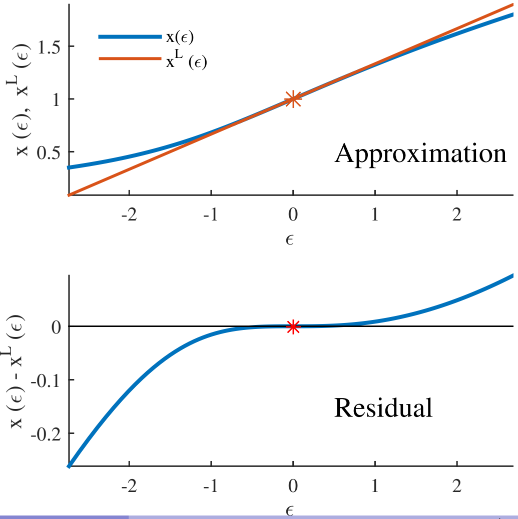
$$f_x(x(\epsilon),\epsilon) = 3x^2 - \epsilon; \quad f_\epsilon(x(\epsilon),\epsilon) = -x$$

- Since $x(0) = 1$, the derivative is

$$x'(0) = -\frac{x(0)}{3x^2(0) - 0} = \frac{1}{3}$$

- The linear approximation is

$$x(\epsilon) \approx 1 + \tfrac{1}{3}\epsilon$$



The economist's problem

- You have an intertemporal model with n endogenous variables $\mathbf{x}_t$, whose dynamics are described by $\mathbf{x}_{t+1} = \Psi(\mathbf{x}_t)$.
- You have found n equilibrium conditions of the form $g(\mathbf{x}_t, \mathbf{x}_{t+1}) = 0$.
- The model is stationary, as such Ψ has a fixed point $\mathbf{x}^*$ (the steady state): $\mathbf{x}^* = \Psi(\mathbf{x}^*)$.
- Problem: How to analyze the dynamics of the model *without* an explicit solution for $\Psi(\mathbf{x})$?

The perturbation method solution

- Solution: Use Taylor's theorem and the implicit function theorem to *approximate* the function Ψ around $\mathbf{x}^*$:
- For example, the first order approximation is

$$\begin{aligned}
\mathbf{x}_{t+1} - \mathbf{x}^* &\approx \left[\nabla f(\mathbf{x}^*)\right]' (\mathbf{x}_t - \mathbf{x}^*) && \text{(Taylor)} \\
&= -\left[D g_{x_{t+1}}(\mathbf{x}^*, f(\mathbf{x}^*))\right]^{-1} D g_{x_t}(\mathbf{x}^*, f(\mathbf{x}^*))(\mathbf{x}_t - \mathbf{x}^*) && \text{(IFT)} \\
&= -\left[D g_{x_{t+1}}(\mathbf{x}^*, \mathbf{x}^*)\right]^{-1} D g_{x_t}(\mathbf{x}^*, \mathbf{x}^*)(\mathbf{x}_t - \mathbf{x}^*) && \text{(FP)}
\end{aligned}$$

Example: Solow model

- To illustrate the procedure of perturbation, we use Solow model
- It has only one dynamic equation, which will make easier to see the logic

Example: Solow model

In the Solow model, capital accumulates according to:

$$0 = g(k_t, k_{t+1}) = sAk_t^\alpha + (1-\delta)k_t - (1+n)k_{t+1}$$

The steady state is

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

Notice that in this example, the solution is trivial: $k_{t+1} = \Psi(k_t) = \frac{sA}{1-n}k_t^\alpha + \frac{1-\delta}{1-n}k_t$. We will use this model to illustrate the technique and to evaluate the quality of the approximation.

Finding the partial derivatives of Solow equation

$g(k_t, k_{t+1}) = sAk_t^\alpha + (1-\delta)k_t - (1+n)k_{t+1}$

$$
\begin{aligned}
&\text{derivative} \ldots && \longrightarrow && \text{evaluated at } k^* \\
g_k &= \alpha s A k_t^{\alpha-1} + (1-\delta) && && \alpha(\delta+n) + 1 - \delta \\
g_{k'} &= -(1+n) && && -1-n \\
g_{kk} &= \alpha(\alpha-1)sAk_t^{\alpha-2} && && \alpha(\alpha-1)(\delta+n)\left(k^*\right)^{-1} \\
g_{kk'} &= 0 && && 0 \\
g_{k'k'} &= 0 && && 0
\end{aligned}
$$

Finding the implicit derivatives on Solow equation

$k_{t+1} = f(k_t)$

Using the implicit function theorem, the first derivative of k_{t+1} with respect to k_t, evaluated at k^*, is

$$f'(k^*) = \frac{\mathrm{d}k'}{\mathrm{d}k} = -\frac{g_k}{g_{k'}} = \frac{\alpha(\delta+n)+1-\delta}{1+n}$$

and the second derivative is

$$\begin{aligned}\frac{\mathrm{d}^2 k'}{\mathrm{d}k^2} &= -\frac{1}{g_{k'}}\left\{g_{kk} + 2g_{kk'}f'(k^*) + g_{k'k'}[f'(k^*)]^2\right\} \\ &= \frac{\alpha(\alpha-1)(\delta+n)}{(1+n)k^*}\end{aligned}$$

First-order approximation to Solow model

In this case

$$k_{t+1} - k^* \approx - \left[Dg_{k_{t+1}}(k^*, k^*)\right]^{-1} Dg_{k_t}(k^*, k^*)(k_t - k^*)$$

$$= \frac{\alpha(\delta + n) + 1 - \delta}{1 + n}(k_t - k^*)$$

$$= \tfrac{1+n-(1-\alpha)(n+\delta)}{1+n}(k_t - k^*)$$

Notice that $\left|\tfrac{1+n-(1-\alpha)(n+\delta)}{1+n}\right| < 1$, so the system is stable.

Second-order approximation to Solow model

We just need to add the quadratic term to the previous approximation

$$k_{t+1} - k^* \approx \frac{1+n-(1-\alpha)(n+\delta)}{1+n}(k_t - k^*) - \frac{\alpha(1-\alpha)(\delta+n)}{2(1+n)k^*}(k_t - k^*)^2$$

Let $\hat{k} = (k - k^*)/k^*$. After dividing both sides by k^*, the approximation becomes

$$\hat{k}_{t+1} \approx \frac{1+n-(1-\alpha)(n+\delta)}{1+n}\,\hat{k}_t - \frac{\alpha(1-\alpha)(\delta+n)}{2(1+n)}\,\hat{k}_t^2$$

This will converge as long as

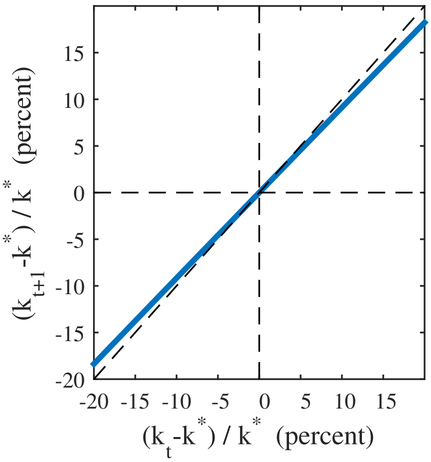
$$-\frac{1}{\alpha} < \hat{k}_t < \frac{2(1+n)-(1-\alpha)(n+\delta)}{\alpha(1-\alpha)(n+\delta)}$$

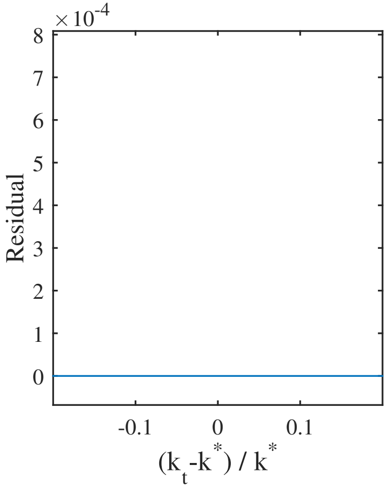
Approximating the adjustment to steady state

Linear and quadratic approximations around $k^* = 0.51$

Parameters:

s	0.10
A	1.00
α	0.40
δ	0.10
n	0.05



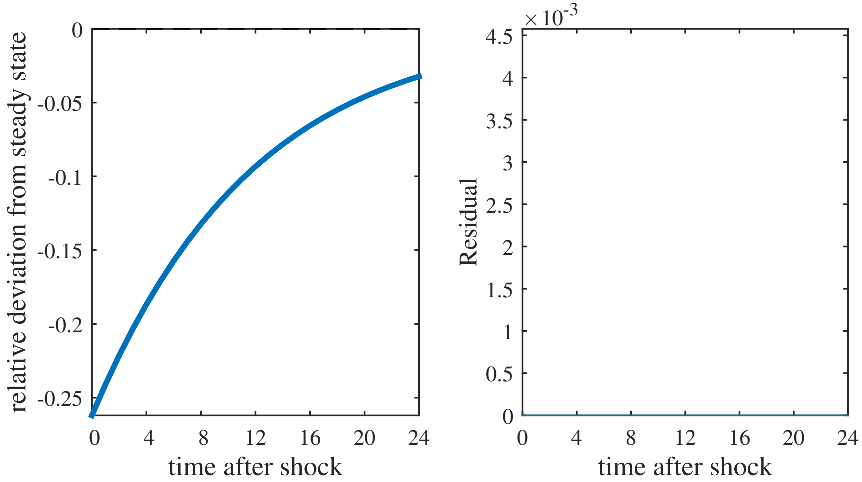


Approximating the impulse response function

Linear and quadratic response to a 20% increase in A

Parameters:

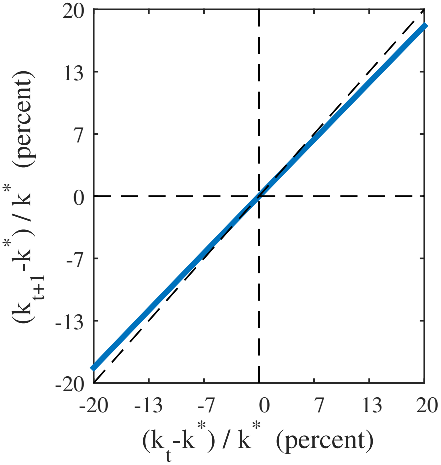
	pre-	post-
s	0.10	0.10
A	1.00	1.20
α	0.40	0.40
δ	0.10	0.10
n	0.05	0.05
k^*	0.51	0.69

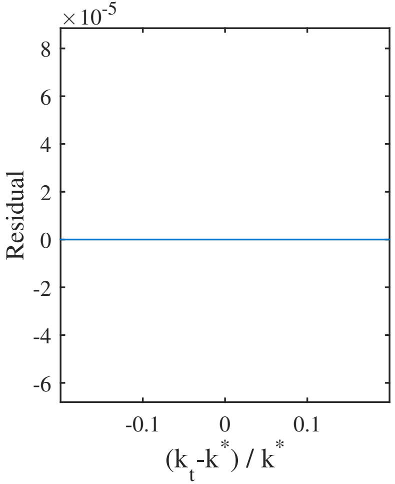


Perturbation vs Chebyshev collocation

Quadratic polynomial approximations:

- Chebyshev over $k_t \in [0.8k^*, 1.2k^*]$, requires knowing $f(k_i)$ at three specific nodes k_i.
- Taylor around $k^* = 0.51$, requires knowing $k^* = f(k^*)$, $f'(k^*)$, and $f''(k^*)$





Example: Ramsey model

The solution of the Ramsey problem is characterized by the equations [1]

$$
\begin{aligned}
0 &= g^1(K_t, C_t, K_{t+1}, C_{t+1}) = K_{t+1} - f(K_t) + C_t \\
0 &= g^2(K_t, C_t, K_{t+1}, C_{t+1}) = u'(C_t) - \beta u'(C_{t+1}) f'(K_{t+1})
\end{aligned}
$$

The perturbation method will allow us to approximate the function Ψ

$$
\begin{bmatrix} K_{t+1} \\ C_{t+1} \end{bmatrix} = \Psi\left(\begin{bmatrix} K_{t+1} \\ C_{t+1} \end{bmatrix}\right)
$$

[1] This example follows the presentation by Heer and Maußner 2009, pp. 77–

Finding the steady state

The steady state is given by

$$0 = g^1(K^*, C^*, K^*, C^*) = K^* - f(K^*) + C^*$$
$$0 = g^2(K^*, C^*, K^*, C^*) = u'(C^*) - \beta u'(C^*) f'(K^*)$$

that is, the steady state k^*, c^* satisfies

$$f'(k^*) = \beta^{-1}$$
$$c^* = f(k^*) - k^*$$

Finding the Jacobian

Let $\mathbf{x}_t = (k_t, c_t)'$. From the equations

$$
\begin{aligned}
0 &= g^1(K_t, C_t, K_{t+1}, C_{t+1}) = K_{t+1} - f(K_t) + C_t \\
0 &= g^2(K_t, C_t, K_{t+1}, C_{t+1}) = u'(C_t) - \beta u'(C_{t+1}) f'(K_{t+1})
\end{aligned}
$$

we find the Jacobian of g evaluated at the steady state

$$
\begin{aligned}
J(\mathbf{x}_t, \mathbf{x}_{t+1}) &= \left[Dg_{x_t} \mid Dg_{x_{t+1}} \right] \\
&= \begin{bmatrix} -f'(k^*) & 1 & 1 & 0 \\ 0 & u''(c^*) & -\beta u'(c^*) f''(k^*) & -\beta u''(c^*) f'(k^*) \end{bmatrix} \\
&= \begin{bmatrix} -\beta^{-1} & 1 & 1 & 0 \\ 0 & u'' & -\beta u' f'' & -u'' \end{bmatrix}
\end{aligned}
$$

The first-order approximation

The first order approximation is

$$\begin{bmatrix} k_{t+1} - k^* \\ c_{t+1} - c^* \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ -\beta u' f'' & -u'' \end{bmatrix}^{-1} \begin{bmatrix} -\beta^{-1} & 1 \\ 0 & u'' \end{bmatrix} \begin{bmatrix} k_t - k^* \\ c_t - c^* \end{bmatrix}$$

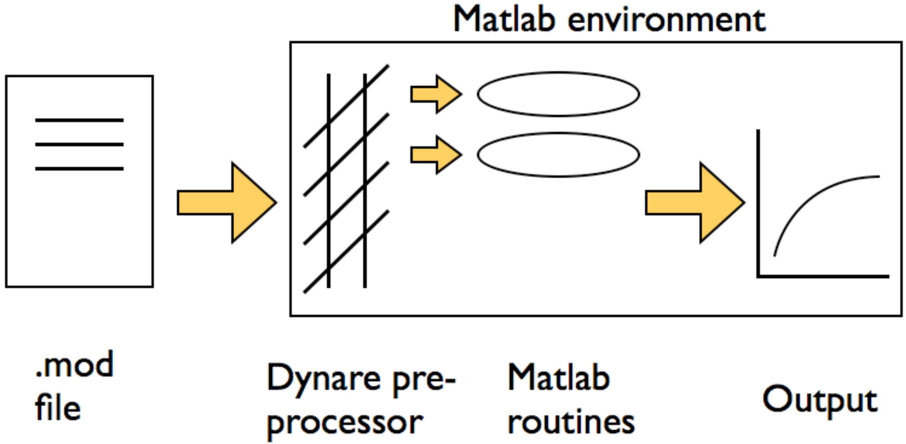
$$= \begin{bmatrix} \beta^{-1} & -1 \\ -\frac{u' f''}{u''} & 1 + \frac{\beta u' f''}{u''} \end{bmatrix} \begin{bmatrix} k_t - k^* \\ c_t - c^* \end{bmatrix}$$

Outline

What is Dynare?

- "Dynare is a powerful and highly customizable engine, with an intuitive front-end interface, to solve, simulate and estimate DSGE models."
- It is a pre-processor and a collection of Matlab routines that has the great advantages of reading DSGE model equations written almost as in an academic paper.
- This not only facilitates the inputting of a model, but also enables you to easily share your code as it is straightforward to read by anyone.

Dynare's workflow



Source: Mancini Griffoli 2013

What is Dynare able to do?

- Compute the steady state of a model
- Compute the solution of deterministic models
- Compute the first and second order approximation to solutions of stochastic models
- Estimate parameters of DSGE models using either a maximum likelihood or a Bayesian approach
- Compute optimal policies in linear-quadratic models

A fundamental distinction

- Dynare can solve both deterministic and stochastic models.
- The distinction hinges on **whether future shocks are known**.
    - In deterministic models, the occurrence of all future shocks is know *exactly* at the time of computing the model's solution.
    - In stochastic models, only the *distribution* of future shocks is known.
- If you only consider a *first-order* linear approximation of the stochastic model, or a linear model, the two cases become practically the same, due to certainty equivalence.

Deterministic vs stochastic models

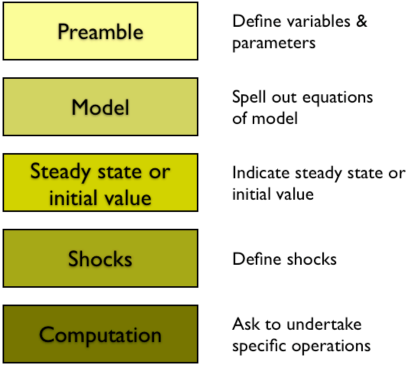
Deterministic models:

- assume full information, perfect foresight, and no uncertainty about shocks.
- shocks can last one or more periods.
- the solution does *not* require linearization: exact path of endogenous variables.
- solution is useful even when economy is far away from steady state.

Stochastic models

- assume that shocks hit today (with a surprise), but thereafter their expected value is zero.
- expected future shocks, or permanent changes in the exogenous variables *cannot* be handled due to the use of Taylor approximations around a steady state.
- solution can be poor when economy is far from steady state.

A .mod file for a stochastic model



Structure of the .mod file

Source: Mancini Griffoli 2013

```
// This is a sample .mod file for Dynare
var y1  y2 ... yn;                    // n endogenous variables
varexo x1  x2 ... xm;                 // m exogenous variables
parameters φ1  φ2 ... φp;             // p parameters
      φ_1 = 0.25;
        ⋮
      φ_p = 0.91;
model;                                // n equations
      eq1 ;
       ⋮
      eq_n;
end;
initval;                              // n + m initial values
      y1 = y1,0;
        ⋮
      x_m = x_m,0;
end;
steady;
shocks;                               // shocks to system
      x_i = x_i,0;
        ⋮
      x_j = x_j,0;
end;
stoch_simul (periods=h);              // stochastic simulation
```

Installing Dynare in Matlab: a warning

- When installing Dynare, you should add Dynare to the Matlab path.
- This can be done by typing **addpath('c:\dynare\4.x.y\matlab')** , where "x.y" refers to the version of Dynare (e.g., the examples in this presentation were done with version 4.3.2)
- Do NOT add all Dynare subfolders in "c:\dynare\4.x.y" to the Matlab path, as doing so will add functions whose names conflict with those of Matlab functions.

Outline

Simulating deterministic models in Dynare

- When the framework is deterministic, Dynare can be used for models with the assumption of perfect foresight.
- Typically, the system is supposed to be in a state of equilibrium before a period '1' when the news of a contemporaneous or of a future shock is learned by the agents in the model.
- The purpose of the simulation is to describe the reaction in anticipation of, then in reaction to the shock, until the system returns to the old or to a new state of equilibrium.
- In most models, this return to equilibrium is only an asymptotic phenomenon, which one must approximate by an horizon of simulation far enough in the future.
- Another exercise for which Dynare is well suited is to study the transition path to a new equilibrium following a permanent shock.

The output from Dynare

oo_.endo_simul The simulated endogenous variables are available in global matrix oo_.endo_simul. This variable stores the result of a deterministic simulation (computed by simul) or of a stochastic simulation (computed by stoch_simul with the periods option or by extended_path). The variables are arranged row by row, in order of declaration (as in M_.endo_names). Note that this variable also contains initial and terminal conditions, so it has more columns than the value of periods option.

oo_.exo_simul This variable stores the path of exogenous variables during a simulation. The variables are arranged in columns, in order of declaration (as in M_.endo_names). Periods are in rows. Note that this convention regarding columns and rows is the opposite of the convention for oo_.endo_simul!

The Solow Model

In the Solow model

- Production: $Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$
- Consumption: $C_t = (1-s)Y_t$
- Capital accum: $K_t = sY_{t-1} + (1-\delta)k_{t-1}$
- Labor growth: $N_t = (1+n)N_{t-1}$

The variables are de-trended: $x_t \equiv \frac{X_t}{N_t}$. Model is:

$$
\begin{aligned}
y_t &= A_t k_{t-1}^{\alpha} \\
c_t &= (1-s)y_t \\
(1+n)k_t &= sy_t + (1-\delta)k_{t-1}
\end{aligned}
$$

The parameters are

$$
\begin{aligned}
\alpha &\in (0,1) \\
\delta &\in (0,1) \\
s &\in (0,1) \\
n &> 0 \\
K_0 &\quad \text{given}
\end{aligned}
$$

The Solow Model: steady state

In steady state:

$$y^* = Ak^{*\alpha}$$

$$c^* = (1-s)y^*$$

$$(1+n)k^* = sy^* + (1-\delta)k^*$$

The steady state values are

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y^* = A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$c^* = (1-s)A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

From model to Dynare: Preamble

Model	Dynare
	// Declare variables and parameters
$c_t \quad y_t \quad k_t$	**var** c y k ;
A_t	**varexo** A;
$\alpha \quad \delta \quad s \quad n$	**parameters** alpha delta s n;
$\alpha = 0.40$	alpha = 0.40;
$\delta = 0.10$	delta = 0.10;
$s = 0.10$	s = 0.10;
$n = 0.05$	n = 0.05;

From model to Dynare: Model

Model	Dynare
	// Declare model's equations
	model;
$y_t = A_t k_{t-1}^{\alpha}$	exp(y) = A*exp(alpha*k(-1));
$c_t = (1-s)y_t$	exp(c) = (1-s)*exp(y);
$(1+n)k_t = sy_t + (1-\delta)k_{t-1}$	(1+n)*exp(k) = s*exp(y(-1)) + (1-delta)*exp(k(-1));
	end;

- Note: writing *exp(x)* instead of x allows to compute impulse response function as *percent* deviation from steady-state.

From model to Dynare: Initial values

Model	Dynare
	// Initial values
	initval;
c^*	c = log(0.675);
y^*	y = log(0.75);
k^*	k = log(0.5);
A	A = 1;
	end;
	steady;

- Note: since model is written in log form, the (approximate) steady state values in this block are also written in log form.
- The command *steady* forces all initial values to the (exact) steady states.

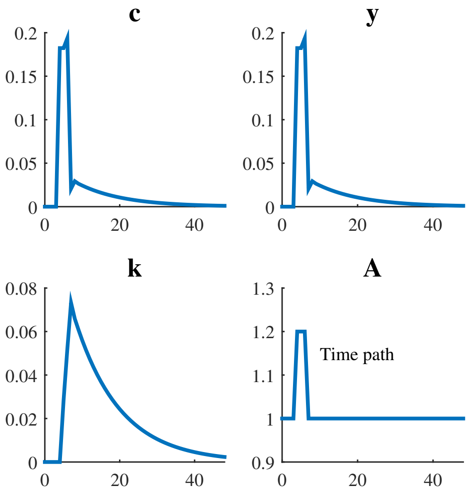
From model to Dynare: Solving the model

Model	Dynare
	// Shocks
	shocks;
$A = 1.2$	var A;
$t = 4 : 6$	periods 1:1;
	values 1.2;
	end;
	steady;
	// Solving
solve!	simul(periods=100);

- We are interested in analyzing the effect of increasing productivity from $A = 1$ to $A = 1.2$ from period 4 to period 6.
- We then simulate the model for 100 periods.

Results: percent deviation from steady state

- Since marginal rate of savings is constant, the response of consumption just mirrors the response on income.
- There is a quick response from capital accumulation to increased productivity; once the shock is gone, capital adjusts slowly towards steady state.
- The initial jump in capital is due to additional savings; the slow adjustment is simply the effect of depreciation.



The Ramsey model

- The following model[2] is a stripped down version of the celebrated model of Kydland and Prescott (1982), who were awarded the Nobel Price in economics 2004 for their contribution to the theory of business cycles and economic policy.
- The model provides an integrated framework for studying economic fluctuations in a growing economy.
- Since it depicts an economy without money it belongs to the class of real business cycle models.
- Similar models appear amongst others in the papers by Hansen (1985), by King, Plosser, and Rebelo (1988a), and by Plosser (1989).

[2] This example is based on Heer and Maußner (2009, pp. 44-46)

The economy

- The economy is inhabited by a representative consumer-producer who derives utility from consumption C_t and leisure $1 - N_t$ and uses labor N_t and capital services K_t to produce output $Y_t = Z_t(A_t N_t)^{1-\alpha} K_t^{\alpha}$.
- Labor augmenting technical progress at the deterministic rate $a > 1$ accounts for output growth: $A_{t+1} = aA_t$.
- Stationary shocks to total factor productivity Z_t induce deviations from the balanced growth path of output: $\ln Z_{t+1} = \rho \ln Z_t + \epsilon_{t+1}$.
- Capital is accumulated according to $K_{t+1} = (1 - \delta)K_t + Y_t - C_t$.

A Ramsey model

The representative agent solves:

$$\max_{C_t, N_t, K_{t+1}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}(1-N_t)^{\theta(1-\eta)}}{1-\eta} \right]$$

subject to

$$\begin{aligned}
K_{t+1} + C_t &= Y_t + (1-\delta)K_t \\
Y_t &= Z_t (A_t N_t)^{1-\alpha} K_t^{\alpha} \\
A_{t+1} &= a A_t \\
\ln Z_{t+1} &= \rho \ln Z_t + \epsilon_{t+1} \\
0 &\leq C_t \\
0 &\leq K_{t+1}
\end{aligned}$$

$$\boxed{\begin{aligned}
a &\geq 1 \\
\alpha &\in (0,1) \\
\beta &\in (0,1) \\
\eta &> \theta/(1+\theta) \\
\theta &\geq 0 \\
\rho &\in (0,1) \\
\sigma &> 0 \\
\\
\epsilon &\sim N(0, \sigma^2) \\
K_0, Z_0 &\quad \text{given}
\end{aligned}}$$

First order conditions

From the Lagrangean

$$\mathfrak{L} := \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}(1-N_t)^{\theta(1-\eta)}}{1-\eta} + \Lambda_t \left[Y_t(N_t, K_t) - C_t - K_{t+1} \right] \right\}$$

we derive the first-order conditions

$$\text{wrt } C_t : \qquad 0 = C_t^{-\eta}(1-N_t)^{\theta(1-\eta)} - \Lambda_t$$

$$\text{wrt } N_t : \qquad 0 = \theta C_t^{1-\eta}(1-N_t)^{\theta(1-\eta)-1} - \Lambda_t(1-\alpha)\frac{Y_t}{N_t}$$

$$\text{wrt } K_{t+1} : \qquad 0 = \Lambda_t - \beta\, \mathbb{E}_t\, \Lambda_{t+1} \left(1 - \delta + \alpha \frac{Y_{t+1}}{K_{t+1}} \right)$$

$$\text{wrt } \Lambda_t : \qquad 0 = K_{t+1} - (1-\delta)K_t + C_t - Y_t$$

First order conditions, stationary version

Substitute out Λ_t to get the system

$$
\begin{aligned}
0 &= \theta C_t - (1-\alpha) Y_t \tfrac{1-N_t}{N_t} \\
0 &= C_t^{-\eta}(1-N_t)^{\theta(1-\eta)} - \beta\,\mathbb{E}_t\, C_{t+1}^{-\eta}(1-N_{t+1})^{\theta(1-\eta)} \left(1-\delta+\alpha\tfrac{Y_{t+1}}{K_{t+1}}\right) \\
0 &= K_{t+1} - (1-\delta)K_t + C_t - Y_t
\end{aligned}
$$

Define $x_t := X_t / A_t$ as a de-trended variable, then FOCs become:

$$
\boxed{
\begin{aligned}
0 &= \theta c_t - (1-\alpha) y_t \tfrac{1-N_t}{N_t} \\
0 &= c_t^{-\eta}(1-N_t)^{\theta(1-\eta)} - \beta a^{-\eta}\,\mathbb{E}_t\, c_{t+1}^{-\eta}(1-N_{t+1})^{\theta(1-\eta)} \left(1-\delta+\alpha\tfrac{y_{t+1}}{k_{t+1}}\right) \\
0 &= a k_{t+1} - (1-\delta)k_t + c_t - y_t \\
0 &= y_t - Z_t N_t^{1-\alpha} k_t^{\alpha}
\end{aligned}
}
$$

The steady state

To find the steady-state, let $Z_t = 1$, drop the expectation operator:

$$0 = \theta c - (1-\alpha) y \frac{1-N}{N}$$

$$0 = 1 - \beta a^{-\eta}\left(1 - \delta + \alpha \frac{y}{k}\right)$$

$$0 = (a + \delta - 1)k + c - y$$

$$0 = y - N^{1-\alpha} k^{\alpha}$$

The (recursive) solution is:

$$N^* = \left\{1 + \frac{\theta}{1-\alpha}\left[\frac{a^\eta - \beta(1-\delta) + \alpha\beta(1-a-\delta)}{a^\eta - \beta(1-\delta)}\right]\right\}^{-1}$$

$$k^* = \left[\frac{\alpha\beta}{a^\eta - \beta(1-\delta)}\right]^{\frac{1}{1-\alpha}} N^*$$

$$y^* = {N^*}^{1-\alpha} {K^*}^{\alpha}$$

$$c^* = y^* - (a + \delta - 1)k^*$$

From model to Dynare: Preamble

Model	Dynare
	// Declare variables and parameters
$c_t \quad y_t \quad k_{t+1} \quad N_t \quad Z_t$	**var** c y k N Z;
k_t	**predetermined_variables** k;
ϵ_{t+1}	**varexo** e;
$a \quad \alpha \quad \beta \quad \delta \quad \eta \quad \theta \quad \rho \quad \sigma$	**parameters** a alpha beta delta eta theta rho sigma;
$a = 1.005$	a = 1.005;
$\alpha = 0.27$	alpha = 0.27;
$\beta = 0.994$	beta = 0.994;
$\delta = 0.011$	delta = 0.011;
$\eta = 2.0$	eta = 2.0;
$\theta = 5.81$	theta = 5.81;
$\rho = 0.90$	rho = 0.90;
$\sigma = 0.0072$	sigma = 0.0072;
$N^* = 0.13$	Nstar = 0.13;

- Dynare assumes that all variables are determined at t.
- But in this model k_t is not decided at time t, but at $t-1$.
- To alert Dynare about this, we need the line *predetermined_variables k;*.

From model to Dynare: Model

Model	Dynare
	// Declare model's equations
	model;
$\theta c_t = (1-\alpha) y_t \frac{1-N_t}{N_t}$	theta*c = (1-alpha)*y*(1-N)/N;
$y_t = a k_{t+1} - (1-\delta) k_t + c_t$	y = a*k(+1) - (1-delta)*k +c;
$y_t = Z_t N_t^{1-\alpha} k_t^{\alpha}$	y = Z*N^(1-alpha)*k^alpha;
$c_t^{-\eta} (1-N_t)^{\theta(1-\eta)} =$ $\beta a^{-\eta} \mathbb{E}_t\, c_{t+1}^{-\eta} (1-N_{t+1})^{\theta(1-\eta)} \left(1-\delta+\alpha \frac{y_{t+1}}{k_{t+1}}\right)$	c^(-eta)*(1-N)^(theta*(1-eta)) = beta*a^(-eta)*c(+1)^(-eta) * (1-N(+1))^(theta*(1-eta)) *(1-delta + alpha*y(+1)/k(+1));
$\ln Z_t = \rho \ln Z_{t-1} + \epsilon_t$	ln(Z) = rho*ln(Z(-1)) + e;
	end;

From model to Dynare: Initial values

Model	Dynare
	// Initial values
	initval;
c^*	c = 0.25;
y^*	y = 0.30;
k^*	k = 3.02;
N^*	N = 0.13;
	Z = 1.00;
$\mathbb{E}\,\epsilon_t$	e = 0;
	end;
	steady;
	check;

- The (approximate) numeric values of the steady state are computed separately using the analytical formulas.
- The command *steady* forces all initial values to the (exact) steady states.
- The command *check* ...

References

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