Dynamic Stochastic General Equilibrium Model

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Introduction

Dynamic Stochastic General Equilibrium (DSGE) models

- DSGE models have become the fundamental tool in current macroeconomic analysis
- They are in common use in academia and in central banks.
- Useful to analyze how economic agents respond to changes in their environment, in a dynamic general equilibrium micro-founded theoretical setting in which all endogenous variables are determined simultaneously.
- Static models and partial equilibrium models have limited value to study how the economy responds to a particular shock.

DSGE: microfoundations + rational expectations

- Modern macro analysis is increasingly concerned with the construction, calibration and/or estimation, and simulation of DSGE models.
- DSGE models start from micro-foundations, taking special consideration of the rational expectation forward-looking economic behavior of agents.

Households

General assumptions about consumers

- There is a representative agent.
- Who is an optimizer: she maximizes a given objective function.
- She lives forever: infinite horizon
- Her happiness depends on consumption C and leisure O.
- The maximization of her objective function is subject to a resource restriction: the budget constraint.

Instant utility

- The instant utility function is

$$u(C, O)$$

- She prefers more consumption and more leisure to less:

$$u_C > 0 \qquad u_O > 0$$

- Higher consumption (and leisure) implies greater utility but at a decreasing rate:

$$u_{CC} < 0 \qquad u_{OO} < 0$$

Expected utility function

- The consumer's happiness depends on the entire path of consumption and leisure that she expects to enjoy:

$$U(C_0, C_1, \dots, C_\infty, O_0, O_1, \dots, O_\infty)$$

- She's impatient: she discounts future utility by β.
- Her utility is time separable.
- Therefore, her expected utility is

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, O_t)$$

Resource ownership

- To define a budget constraint we must introduce property rights.
- Here,we assume that the consumer is the owner of production factors: capital K and L labor.
- L comes from the available endowment of time, which we normalize to 1. Because time cannot be accumulated, labor decisions will be static.
- K is accumulated through investment, which in turn depends on savings.
- Consumer also owns the firm.

The budget constraint

- Household income comes from renting both productive factors to the production sector, at given rental prices.
- Household can do two things with these earnings: expend it in consumption or save it.
- Then, the budget constraint is

$$P_t\left(C_t + S_t\right) \leq W_t L_t + R_t K_t + \Pi_t$$

where

$$\begin{aligned} P_t &= \text{price of consumption good} & S_t &= \text{savings} \\ R_t &= \text{user cost of capital} & W_t &= \text{wage} \\ \Pi_t &= \text{firm's profits } (= \text{dividends}) \end{aligned}$$

- Since there is no money, we normalize $P_t = 1 \quad \forall t$.

Resource constraints

- Since time is spent either working or in leisure:

$$O_t + L_t = 1 \qquad \forall t$$

- Given this constraint, in what follows we write the instant utility function as:

$$u(C, 1 - L)$$

- Because capital deteriorates over time, its accumulation is subject to depreciation rate δ:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

The financial sector

- To keep things simple, we assume that there is a competitive sector that transforms savings directly into investment without any cost.
- Thus

$$S_t = I_t$$

- Combining this assumption with the budget constraint and the capital accumulation equation, the consumer is constraint by

$$\begin{aligned} C_t &\leq W_t L_t + R_t K_t + \Pi_t - S_t \\ &\leq W_t L_t + R_t K_t + \Pi_t - I_t \\ &\leq W_t L_t + R_t K_t + \Pi_t + (1-\delta)K_t - K_{t+1} \\ &\leq W_t L_t + \Pi_t + (1 + R_t - \delta)K_t - K_{t+1} \end{aligned}$$

The consumer problem

The consumer problem is to maximize her lifetime utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, 1 - L_t)$$

subject to the budget constraint*

$$C_t = W_t L_t + \Pi_t + (1 + R_t - \delta) K_t - K_{t+1} \qquad \forall t = 0, 1, \dots$$

where K_0 is predetermined.

*We impose equality because $u_C > 0$.

The consumer problem: dynamic programming

- The consumer problem is recursive, so we can represent it by a Bellman equation.
- Current capital is the state variable, next capital and labor are the policy variables.
- Then we write

$$V(K) = \max_{K', L} \left\{ u(C, 1-L) + \beta\, \mathbb{E}\, V(K') \right\}$$

subject to the budget constraint

$$C = WL + \Pi + (1 + R - \delta)K - K'$$

The consumer problem: solution

- The FOCs are:

$$u_O = W u_C \qquad \text{(wrt labor)}$$
$$u_C = \beta\, \mathbb{E}\, V'(K') \qquad \text{(wrt capital)}$$

- The envelope condition is

$$V'(K) = (1 + R - \delta)\, u_C$$

- Therefore, the Euler equation is

$$u_C = \beta\, \mathbb{E}\left[(1 + R' - \delta) u_{C'}\right]$$

Consumer optimization: In summary

- For the numerical solution of the model, we assume that

$$u\left(C_t, 1-L_t\right) = \gamma \ln C_t + (1-\gamma)\ln(1-L_t)$$

- Therefore, the solution of the consumer problem requires

$$\begin{aligned}
1 &= \beta\,\mathbb{E}\left[(1+R_{t+1}-\delta)\frac{C_t}{C_{t+1}}\right] \\
C_t &= \frac{\gamma}{1-\gamma} W(1-L_t) \\
K_{t+1} &= (1-\delta)K_t + I_t
\end{aligned}$$

Firms

The firms

- Firms produce goods and services the households will consume of save.
- To do this, they transform capital K and labor L into final output.
- They rent these factors from households.

Production function

- Technology is described by the aggregate production function

$$Y_t = A_t F\left(K_t, L_t\right)$$

where Y_t is aggregate output and A_t is total factor productivity (TFP).

- Production increases with inputs...

$$F_K > 0 \qquad\qquad F_L > 0$$

- ...but marginal productivity of each factor is decreasing :

$$F_{KK} < 0 \qquad\qquad F_{LL} < 0$$

Production function (cont'n)

We assume that

- Production has constant returns to scale:

$$A_t F\left(\lambda K_t, \lambda L_t\right) = \lambda Y_t$$

- Both factors are indispensable for production

$$A_t F\left(0, L_t\right) = 0 \qquad\qquad A_t F\left(K_t, 0\right) = 0$$

- Production satisfies the Inada conditions

$$\lim_{K\to 0} F_K = \infty \qquad\qquad \lim_{L\to 0} F_L = \infty$$
$$\lim_{K\to \infty} F_K = 0 \qquad\qquad \lim_{L\to \infty} F_L = 0$$

The firm's problem: static optimization

- Firms maximize profits, subject to the technological constraint.

$$\max_{K_t, L_t} \Pi_t = Y_t - W_t L_t - R_t K_t$$

$$\text{s.t.} \quad Y_t = A_t F\left(K_t, L_t\right)$$

or simply

$$\max_{K_t, L_t} A_t F\left(K_t, L_t\right) - W_t L_t - R_t K_t$$

The firm's problem: solution

- The FOCs are:

$$W_t = A_t F_L\left(K_t, L_t\right) \qquad \text{(wrt labor)}$$
$$R_t = A_t F_K\left(K_t, L_t\right) \qquad \text{(wrt capital)}$$

that is, the relative price of productive factors equals their marginal productivity.

Side note:

Euler's theorem

Euler's theorem

Let $f(x)$ be a $\mathcal{C}^1$ homogeneous function of degree k on $\mathbb{R}^n_+$. Then, for all x,

$$x_1 \frac{\partial f}{\partial x_1}(x) + x_2 \frac{\partial f}{\partial x_2}(x) + \cdots + x_n \frac{\partial f}{\partial x_n}(x) = kf(x)$$

Proof

Because $f(x)$ is homogeneous of degree k on $\mathbb{R}^n_+$:

$$f(\lambda x) = f\left(\lambda x_1, \lambda x_2, \ldots, \lambda x_n\right) = \lambda^k f(x)$$

Then, since $f(x)$ is $\mathcal{C}^1$, for all x we take derivative with respect to λ:

$$x_1 \frac{\partial f}{\partial \lambda x_1}(x) + x_2 \frac{\partial f}{\partial \lambda x_2}(x) + \cdots + x_n \frac{\partial f}{\partial \lambda x_n}(x) = k\lambda^{k-1} f(x)$$

Finally, setting $\lambda = 1$ we get the result.

The firm's profits

- Since F is homogeneous of degree one (constant returns to scale), Euler's theorem implies

$$[A_t F_K(K_t, L_t)]\, K_t + [A_t F_L(K_t, L_t)]\, L_t = Y_t$$

- Substitute FOCs from firms problem:

$$R_t K_t + W_t L_t = Y_t$$

- and therefore optimal profits will equal zero:

$$\Pi_t = Y_t - R_t K_t - W_t L_t = 0$$

The total factor productivity

- The TFP A_t follows a first-order autoregressive process:

$$\ln A_t = (1-\rho)\ln \bar{A} + \rho \ln A_{t-1} + \epsilon_t$$

  where the productivity shock ϵ_t is a Gaussian white noise process:

$$\epsilon_t \sim N(0, \sigma^2)$$

- This assumption led to the birth of the Real Business Cycle (RBC) literature.

The total factor productivity (cont'n)

- The TFP process can also be written

$$\ln A_t - \ln \bar{A} = \rho\left(\ln A_{t-1} - \ln \bar{A}\right) + \epsilon_t$$

- In equilibrium, $A_t = \bar{A}$.
- Productivity shocks cause persistent deviations in productivity from its equilibrium value:

$$\frac{\partial\left(\ln A_{t+s} - \ln \bar{A}\right)}{\partial \epsilon_t} = \rho^{s-1} > 0$$

as long as $\rho > 0$.
- Although persistent, the effect of a shock is not permanent

$$\lim_{s\to\infty} \frac{\partial\left(\ln A_{t+s} - \ln \bar{A}\right)}{\partial \epsilon_t} = \rho^{s-1} = 0$$

Firm optimization: In summary

- For the numerical solution of the model, we assume that

$$A_t F(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha} \qquad \text{and} \qquad \bar{A} = 1$$

- Therefore, the solution of the firm problem requires

$$
\begin{aligned}
W_t &= (1-\alpha) A_t \left(\frac{K_t}{L_t}\right)^{\alpha} = (1-\alpha)\frac{Y_t}{L_t} \\
R_t &= \alpha A_t \left(\frac{L_t}{K_t}\right)^{1-\alpha} = \alpha \frac{Y_t}{K_t} \\
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
\ln A_t &= \rho \ln A_{t-1} + \epsilon_t
\end{aligned}
$$

The competitive equilibrium

Putting the agents together

- The equilibrium of this models depends on the interaction of consumers and firms.
- Households decide how much to consume C_t, to invest (save) $I_t = S_t$, and to work L_t, with the objective of maximizing their happiness, taking as given the prices of inputs.
- Firms decide how much to produce Y_t, by hiring capital K_t and labor L_t, given the prices of production factors.
- Since both agents take all prices as given, this is a competitive equilibrium.

The competitive equilibrium

The competitive equilibrium for this economy consists of

1. A pricing system for W and R
2. A set of values assigned to Y, C, I, L and K.

such that

1. given prices, the consumer optimization problem is satisfied;
2. given prices, the firm maximizes its profits; and
3. all markets clear at those prices.

Competitive Equilibrium

The competitive equilibrium for this economy consists of prices W_t and R_t, and quantities A_t, Y_t, C_t, I_t, L_t and K_{t+1} such that:

$$
\begin{aligned}
1 &= \beta\,\mathbb{E}\left[(1+R_{t+1}-\delta)\frac{C_t}{C_{t+1}}\right] \\
C_t &= \frac{\gamma}{1-\gamma}W(1-L_t) \\
K_{t+1} &= (1-\delta)K_t + I_t \\
W_t &= (1-\alpha)\frac{Y_t}{L_t} \\
R_t &= \alpha\frac{Y_t}{K_t} \\
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
\ln A_t &= \rho \ln A_{t-1} + \epsilon_t \\
Y_t &= C_t + I_t
\end{aligned}
$$

- First 3 equations characterize solution of consumer problem
- Next 3 equations characterize solution of firm problem
- Next equation governs dynamic of TFP
- Last equation implies equilibrium in goods markets
- Equilibrium in factor markets is implicit: we use same K, L in consumer and firm problems
- Later, we use these 8 equations in IRIS to solve and simulate the model.

Welfare theorems

If there are no distortions such as (distortionary) taxes or externalities, then

1^{st} Welfare Theorem The competitive equilibrium characterized in last slide is Pareto optimal

2^{nd} Welfare Theorem For any Pareto optimum a price system W_t, R_t exists which makes it a competitive equilibrium

The central planning equilibrium

The central planner

- An alternative setting to a competitive market environment is to consider a centrally planned economy
- The central planner makes all decisions in the economy.
- Objective: the joint maximization of social welfare
- Prices have no role in this setting.

The central planner problem

The central planner problem is to maximize social welfare

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, 1 - L_t)$$

subject to the constraints $\qquad \forall t = 0, 1, \dots$

$$\begin{aligned} C_t + I_t &= Y_t && \text{(resource constraint)} \\ Y_t &= A_t F(K_t, L_t) && \text{(technology constraint)} \\ K_{t+1} &= (1-\delta) K_t + I_t && \text{(capital accumulation)} \end{aligned}$$

where K_0 is predetermined. The three constraints can be combined into

$$C_t + K_{t+1} = A_t F(K_t, L_t) + (1-\delta) K_t$$

The central planner problem: dynamic programming

- The central planner problem is recursive too, so we can represent it by a Bellman equation.
- "Current capital" is the state variable, "next capital" and "labor" are the policy variables.
- Then we write

$$V(K) = \max_{K', L} \left\{ u(C, 1 - L) + \beta\, \mathbb{E}\, V(K') \right\}$$

subject to the constraint

$$C = AF(K, L) + (1 - \delta)K - K'$$

The central planner problem: solution

- The FOCs are:

$$u_O = u_C A F_L(K, L) \qquad \text{(wrt labor)}$$
$$u_C = \beta \, \mathbb{E} \, V'(K') \qquad \text{(wrt capital)}$$

- The envelope condition is

$$V'(K) = \left[A F_K(K, L) + 1 - \delta \right] u_C$$

- Therefore, the Euler equation is

$$u_C = \beta \, \mathbb{E} \left\{ \left[A F_{K'}(K', L') + 1 - \delta \right] u_{C'} \right\}$$

Central planner optimization: In summary

- For the numerical solution of the model, we assume again that

$$\begin{aligned} u\left(C_t, 1-L_t\right) &= \gamma \ln C_t + (1-\gamma)\ln(1-L_t) \\ F(K_t, L_t) &= K_t^{\alpha} L_t^{1-\alpha} \end{aligned}$$

- Therefore, the solution of the central planner problem requires

$$\begin{aligned} 1 &= \beta\, \mathbb{E}\left[\left(\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right) \frac{C_t}{C_{t+1}}\right] \\ C_t &= \frac{\gamma}{1-\gamma}(1-\alpha)\frac{Y_t}{L_t}(1-L_t) \\ K_{t+1} &= (1-\delta)K_t + I_t \end{aligned}$$

The central planning equilibrium

The central planning equilibrium

The central planning equilibrium for this economy consists quantities A_t, Y_t, C_t, I_t, L_t and K_{t+1} such that:

$$1 = \beta\,\mathbb{E}\left[\left(\alpha\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right)\frac{C_t}{C_{t+1}}\right]$$

$$\frac{C_t}{1-L_t} = \frac{\gamma}{1-\gamma}(1-\alpha)\frac{Y_t}{L_t}$$

$$K_{t+1} = (1-\delta)K_t + I_t$$

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

$$\ln A_t = \rho \ln A_{t-1} + \epsilon_t$$

$$Y_t = C_t + I_t$$

- These equations characterize solution of the social planner problem
- There are no market equilibrium conditions, because there are no markets

Central planner vs. competitive market equilibria

- The solution under a centrally planned economy is exactly the same as under a competitive market.
- This is because there are no distortions in our model that alters the agents' decisions regarding the efficient outcome.
- Only difference: In central planner setting there are no markets for production factors, and therefore no price for factors either.

The steady state

The steady state

- The steady state refers to a situation in which, in the absence of random shocks, the variables are constant from period to period.
- Since there is no growth in our model, it is stationary, and therefore it has a steady state.
- We can think of the stead state as the long term equilibrium of the model.
- To calculate the steady state, we set all shocks to zero and drop time indices in all variables.

Computing the steady state for the competitive equilibrium

In this case, the steady state consists of prices $\bar{W}$ and $\bar{R}$, and quantities $\bar{A}$, $\bar{Y}$, $\bar{C}$, $\bar{I}$, $\bar{L}$ and $\bar{K}$ such that:

$$1 = \beta(1 + \bar{R} - \delta) \qquad \bar{C} = \frac{\gamma}{1-\gamma}\bar{W}(1-\bar{L})$$

$$\bar{I} = \delta\bar{K} \qquad \bar{W} = (1-\alpha)\frac{\bar{Y}}{\bar{L}}$$

$$\bar{R} = \alpha\frac{\bar{Y}}{\bar{K}} \qquad \bar{Y} = \bar{A}\bar{K}^{\alpha}\bar{L}^{1-\alpha}$$

$$\bar{A} = 1 \qquad \bar{Y} = \bar{C} + \bar{I}$$

IRIS

IRIS

- IRIS is a free, open-source toolbox for macroeconomic modeling and forecasting in Matlab®, developed by the IRIS Solutions Team since 2001.
- In a user-friendly command-oriented environment, IRIS integrates core modeling functions (flexible model file language, tools for simulation, estimation, forecasting and model diagnostics) with supporting infrastructure (time series analysis, data management, or reporting).
- It can be downloaded from Github.

Solving the model with IRIS

- To solve the model, one creates two files
  - **.model** Here we describe the model: declare its variables, parameters, and equations
  - **.m** this is a regular MATLAB file. Here we load the model, solve it, and analyze it.
- The code presented here is based on Torres (2015, pp.51-52), which was written to be used with DYNARE instead of IRIS.

The .model file: Define variables

- The .model file is a text file, where we declare (usually) four sections:
  - `!transition_variables`
  - `!transition_shocks`
  - `!parameters`
  - `!transition_equations`
- Although not required, using `'labels'` greatly improves readability.

IRIS, part 1

```
!transition_variables
'Income' Y
'Consumption' C
'Investment' I
'Capital' K
'Labour' L
'Wage' W
'Real interest rate' R
'Productivity' A

!transition_shocks
'Productivity shock' e
```

Model calibration

For the model to be completely computationally operational, a value must be assigned to the parameters.

Parameter	Definition	Value
α	Marginal product of capital	0.35
β	Discount factor	0.97
γ	Preference parameter	0.40
δ	Depreciation rate	0.06
ρ	TFP autoregressive parameter	0.95
σ	TFP standard deviation	0.01

The .model file: Define and calibrate parameters

- In the `!parameters` section, we declare all parameters (so we can use them later in the equations)
- Optionally, we can calibrate them here, (otherwise we do it in the .m file)

IRIS, part 2

```
!parameters
'Income share of capital' alpha = 0.35
'Discount factor' beta = 0.97
'Preferences parameter' gamma = 0.40
'Depreciation rate' delta = 0.06
'Autorregresive parameter' rho = 0.95
```

The .model file: Specify the model equations

- Equations are separated by semicolon
- Lags are indicated by `{-n}`, leads by `{+n}`

IRIS, part 3

```
!transition_equations
'Consumption vs. leisure choice'
C = (gamma/(1-gamma))*(1-L)*(1-alpha)*Y/L;
'Euler equation'
1 = beta * ((C/C{+1}) * (R{+1} + (1-delta)));
'Production function'
Y = A*(K{-1}^alpha) * (L^(1-alpha));
'Capital accumulation'
K = I + (1 - delta) * K{-1};
'Investment equals savings'
I = Y - C;
'Labor demand'
W = (1-alpha) * A * (K{-1} / L)^alpha;
'Capital demand'
R = alpha * A * (L / K{-1})^(1-alpha);
'Productivity AR(1) process'
log(A) = rho * log(A{-1}) + e;
```

The .m file: Working with the model

- The .m file is a Matlab file, where we work with the model
- To work with IRIS, we need to add it to the path using `addpath`
- It is recommended to start with a clean session
- We read the model using `model`

Matlab, part 1

```
clear all
close all
clc
addpath C:\IRIS
irisstartup()

%% READ MODEL FILE
m = model('torres-
    chapter2.model');
```

The .m file: Finding the steady state

- IRIS uses the `sstate` command to look for the steady state
- To use it, we have to guess initial values, which we `assign` to the model, starting with the initial parameters in `get(m,'params')`

Matlab, part 2

```
%% INITIAL VALUES
P = get(m, 'params');
P.Y = 1;
P.C = 0.8;
P.L = 0.3;
P.K = 3.5;
P.I = 0.2;
P.W = (1-P.alpha)*P.Y/P.L;
P.R = P.alpha * P.Y/P.K;
P.A = 1;

%% STEADY STATE
m = assign(m, P);
m = sstate(m, 'blocks=',true);
chksstate(m)
get(m,'sstate')
```

Steady states: results

We find that the steady state is given by

Variable	Definition	Value	Ratio to $\bar{Y}$
$\bar{Y}$	Output	0.7447	1.000
$\bar{C}$	Consumption	0.5727	0.769
$\bar{I}$	Investment	0.1720	0.231
$\bar{K}$	Capital	2.8665	3.849
$\bar{L}$	Labor	0.3604	-
$\bar{R}$	Capital rental price	0.0909	-
$\bar{W}$	Real Wage	1.3431	-
$\bar{A}$	TFP	1.0000	-

The .m file: Solving and simulating the model

- IRIS uses the `solve` and `simulate` commands to get the solution and run simulations of the model.
- Here, we simulate the impact of an unanticipated 10% increase in total factor productivity:

$$\ln A_t = 0.95 \ln A_{t-1} + \epsilon_t$$

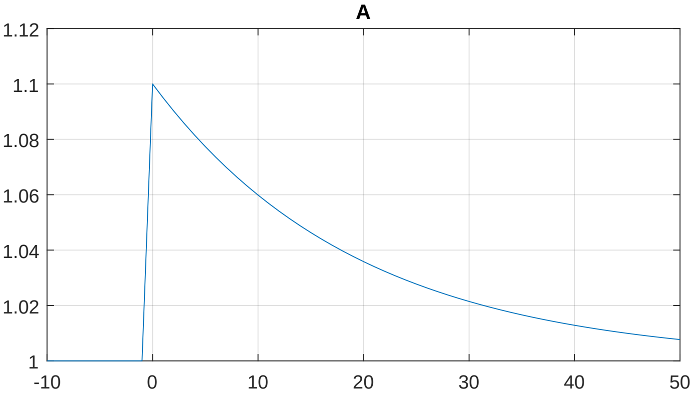
- Notice how persistent the shock is.

Matlab, part 3

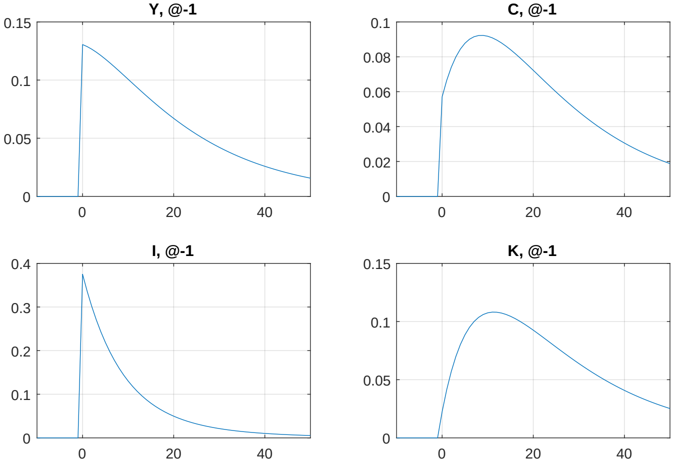
```
%% SOLUTION
m = solve(m);

%% SIMULATE PRODUCTIVITY SHOCK
tt = -10:50; %time range
tshock=0; % shock date
d = sstatedb(m, tt);
d.e(0) = 0.10;
s = simulate(m, d, tt,'
    Anticipate=',false);
```

The .m file: Solving and simulating the model

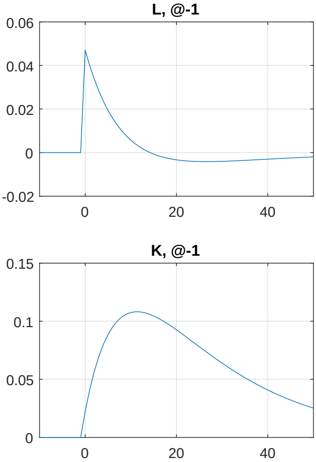


Responses of endogenous variables to productivity shock



Relative deviations respect to pre-shock values

Responses of endogenous variables to productivity shock



Relative deviations respect to pre-shock values

References I

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