Dynamic programming

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Introduction

About this lecture

- We study how to use Bellman equations to solve dynamic programming problems.
- We consider a consumer who wants to maximize his lifetime consumption over an infinite horizon, by optimally allocating his resources through time. Two alternative models:
  1. the consumer uses a financial instrument (say a bank deposit without overdraft limit) to smooth consumption;
  2. the consumer has access to a production technology and uses the level of capital to smooth consumption.
- To keep matters simple, we assume:
  - a logarithmic instant utility function;
  - there is no uncertainty.
- To start, we review some math that we'll need later.

Static optimization

- Optimization is a predominant theme in economic analysis.
- For this reason, the classical calculus methods of finding free and constrained extrema occupy an important place in the economist's everyday tool kit.
- Useful as they are, such tools are applicable only to static optimization problems.
- The solution sought in such problems usually consists of a single optimal magnitude for every choice variable.
- It does not call for a schedule of optimal sequential action.

Dynamic optimization

- In contrast, a dynamic optimization problem poses the question of what is the optimal magnitude of a choice variable in each period of time within the planning period.
- It is even possible to consider an infinite planning horizon.
- The solution of a dynamic optimization problem would thus take the form of an optimal time path for every choice variable, detailing the best value of the variable today, tomorrow, and so forth, till the end of the planning period.

Basic ingredients

A simple type of dynamic optimization problem would contain the following basic ingredients:

1. a given initial point and a given terminal point;
2. a set of admissible paths from the initial point to the terminal point;
3. a set of path values serving as performance indices (cost, profit, etc.) associated with the various paths; and
4. a specified objective-either to maximize or to minimize the path value or performance index by choosing the optimal path.

Alternative approaches to dynamic optimization

To find the optimal path, there are three major approaches:

1. the calculus of variations, dating back to the late 17th century, it works about variations in the **state** path.
2. optimal control theory, the problem is viewed as having both a state and a **control** path, focusing on variations of the control path.
3. dynamic programming, which embeds the control problem in a family of control problems, focusing on the optimal value of the problem (**value function**).

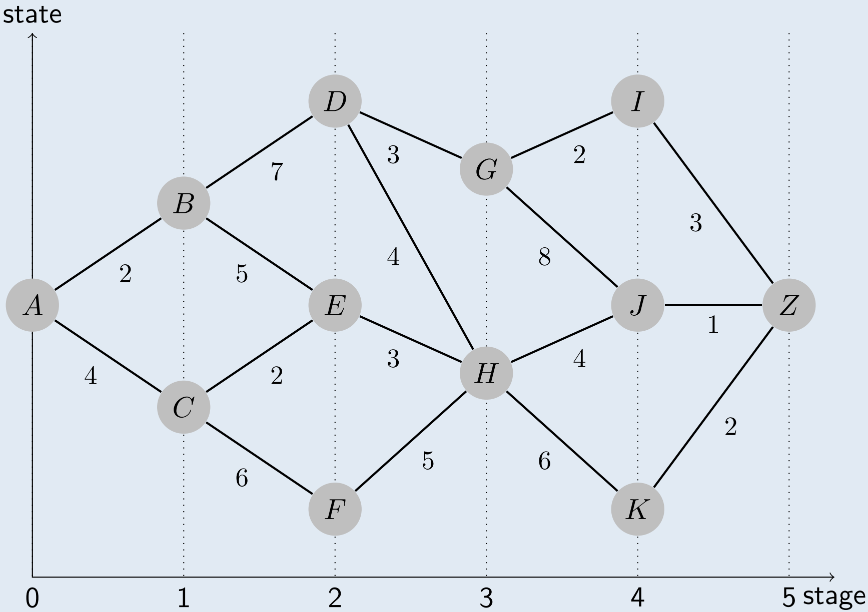
Salient features of dynamic optimization problems

- Although dynamic optimization is mostly couched in terms of a sequence of time, it is also possible to envisage the planning horizon as a sequence of stages in an economic process.
- In that case, dynamic optimization can be viewed as a problem of multistage decision making.
- The distinguishing feature, however, remains the fact that the optimal solution would involve more than one single value for the choice variable.
- The multistage character of dynamic optimization can be illustrated with a simple discrete example.

Example 1:

Multistage decision making

- Suppose that a firm engages in transforming a certain substance from an initial state A (raw material state) into a terminal state Z (finished product state) through a five-stage production process.
- In every stage, the firm faces the problem of choosing among several possible alternative subprocesses, each entailing a specific cost.
- The question is: How should the firm select the sequence of subprocesses through the five stages in order to minimize the total cost?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Basics of dynamic programming

The principle of optimality

The dynamic programming approach is based on the principle of optimality (Bellman, 1957)

> *An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

Why dynamic programming?

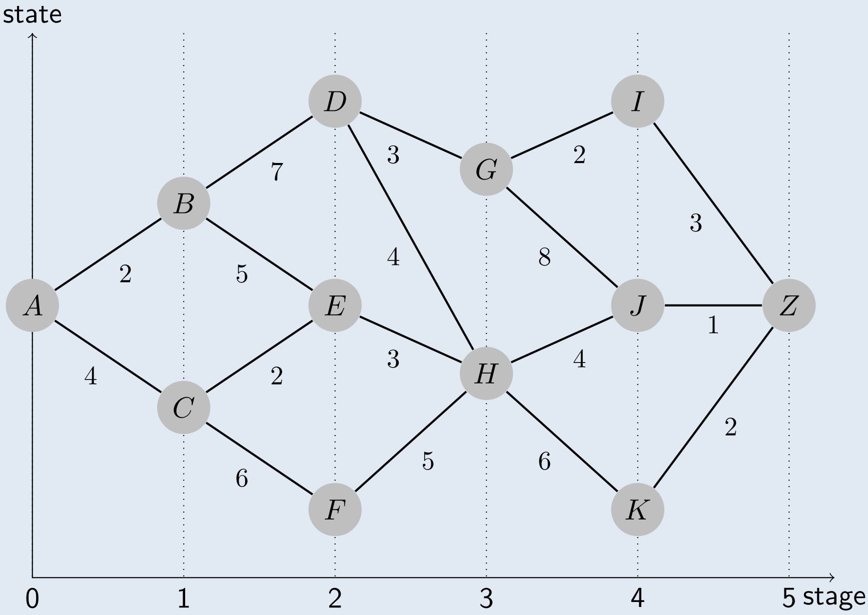
Dynamic programming is a very attractive method for solving dynamic optimization problems because

- it offers backward induction, a method that is particularly amenable to programmable computers, and
- it facilitates incorporating uncertainty in dynamic optimization models.

Example 2:

Solving the problem in example 1

- Let's use dynamic programming to solve example 1.
- Before doing so, let's use this problem to introduce some important concepts:
    - choice and state variables
    - reward and transition functions
    - the value function
    - the Bellman equation
    - the policy function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5 stage

Choice variable In a given state s, firm can choose x among subprocess $\{1, 2, \dots, n\}$.

The reward function $r(x, s)$ returns the cost incurred by choosing a specific subprocess x when state is s. For example

$$r(1, A) = 2 \qquad r(1, G) = 2$$
$$r(2, A) = 4 \qquad r(2, G) = 8$$

The transition function $g(x, s)$ returns the state s' reached in next stage if current state is s and current choice is x. E.g.

$$g(1, A) = B \qquad g(1, G) = I$$
$$g(2, A) = C \qquad g(2, G) = J$$

In this example, we can also think of the choice variable as deciding what state to go next: $x = s'$.

The reward function $r(s, s')$ returns the cost of going from one state to the next. Examples

$$r(A, B) = 2 \qquad r(G, I) = 2$$
$$r(A, C) = 4 \qquad r(G, J) = 8$$

The transition function $g(s, s')$ is now very simple. E.g.

$$g(A, B) = B \qquad g(G, I) = I$$
$$g(A, C) = C \qquad g(G, J) = J$$

- **Objective** Select a sequence of subprocesses through the five

  stages in order to minimize the total cost?
- A sequence is a function: for each stage t, it returns the state of the firm process, $s_t \equiv s(t)$
- Then, total cost is a functional: a function that depends on other function!
- For example

$$c(ABDGIZ) = 2 + 7 + 3 + 2 + 3 = 17 = c(11111)$$
$$c(ABEHKZ) = 2 + 5 + 3 + 6 + 2 = 18 = c(12121)$$

- In general, the objective function is

$$\sum_{t=0}^{4} r(s_t, x_t) \quad \text{subject to} \quad s_{t+1} = g(x_t, s_t), \quad t = 0, \dots, 4$$

- **The value function $V_t(s_t)$** measures the best result that can be achieved in stage t given the current state s_t.

$$V_t(s_t) = \min_{x_0,\dots,x_4} \sum_{t=0}^{4} r(s_t, x_t) \quad \text{s.t.} \quad s_{t+1} = g(x_t, s_t), \quad t = 0, \dots, 4$$

- **The policy function $h_t(s_t)$** returns the best choice that can be made in current state s_t and stage t.

$$h_t(s_t) = \operatorname*{argmin}_{x_0,\dots,x_4} \sum_{t=0}^{4} r(s_t, x_t) \quad \text{s.t.} \quad s_{t+1} = g(x_t, s_t), \quad t = 0, \dots, 4$$

The Bellman equation is based on the principle of optimality,

which for the problem implies

$$V_t(s_t) = \min_{x_t} \{r(x_t, s_t) + V_{t+1}(s_{t+1})\} \quad \text{s.t.} \quad s_{t+1} = g(x_t, s_t)$$
$$V_t(s_t) = \min_{x_t} \{r(x_t, s_t) + V_{t+1}\left(g(x_t, s_t)\right)\}$$

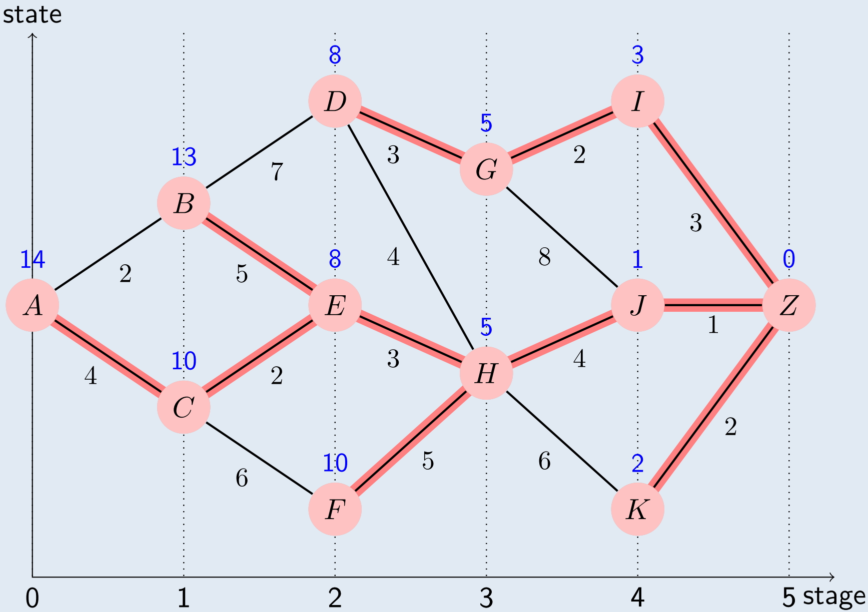
Notice that, by definition of the policy function,

$$V_t(s_t) = r\left[h_t(s_t), s_t\right] + V_{t+1}\left[g\left(h_t(s_t), s_t\right)\right]$$

Let's use the Bellman equation to find the minimum cost of production:

$$V_t(s_t) = \min_{x_t} \left\{ r(x_t, s_t) + V_{t+1}\left(g(x_t, s_t)\right) \right\}$$

- Starting from the terminal state Z: cost to complete product is 0.
- Find the minimum cost from all stage-4 states until product completion.
- Find the minimum cost from all stage-3 states until product completion, taking into account the minimum costs from stage-4 states onward.
- Iterate until reaching stage-0 state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Stage	State		Value	Policy
5	Z	$V_5(Z) = 0$		$h_5(Z) = Z$
4	K	$V_4(K) = \min\{2 + V_5(Z)\} = 2$		$h_4(K) = Z$
	J	$V_4(J) = \min\{1 + V_5(Z)\} = 1$		$h_4(J) = Z$
	I	$V_4(I) = \min\{3 + V_5(Z)\} = 3$		$h_4(I) = Z$
3	H	$V_3(H) = \min\{4 + V_4(J),\ 6 + V_4(K)\} = 5$		$h_3(H) = J$
	G	$V_3(G) = \min\{2 + V_4(I),\ 8 + V_4(J)\} = 5$		$h_3(G) = I$
2	F	$V_2(F) = \min\{5 + V_3(H)\} = 10$		$h_2(F) = H$
	E	$V_2(E) = \min\{3 + V_3(H)\} = 8$		$h_2(E) = H$
	D	$V_2(D) = \min\{3 + V_3(G),\ 4 + V_3(H)\} = 8$		$h_2(D) = G$
1	C	$V_1(C) = \min\{2 + V_2(E),\ 6 + V_1(F)\} = 10$		$h_1(C) = E$
	B	$V_1(B) = \min\{7 + V_2(D),\ 5 + V_2(E)\} = 13$		$h_1(B) = E$
0	A	$V_0(A) = \min\{2 + V_1(B),\ 4 + V_1(C)\} = 14$		$h_0(A) = C$

Dynamic programming: formal setup

Dynamic Programming: the basics

We now introduce basic ideas and methods of dynamic programming (Ljungqvist and Sargent 2004)

- basic elements of a recursive optimization problem
- the Bellman equation
- methods for solving the Bellman equation
- the Benveniste-Scheikman formula

Sequential problems

- Let $\beta \in (0,1)$ be a discount factor.
- We want to choose an infinite sequence of "controls" $\{x_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t r(s_t, x_t) \tag{1}$$

  subject to $s_{t+1} = g(s_t, x_t)$, with $s_0 \in \mathbb{R}$ given.
- We assume that $r(s_t, x_t)$ is a concave function and that the set $\{(s_{t+1}, s_t) : s_{t+1} \leq g(s_t, x_t), x_t \in \mathbb{R}\}$ is convex and compact.

Dynamic programming seeks a time-invariant policy function h mapping the state s_t into the control x_t, such that the sequence $\{x_t\}_{t=0}^{\infty}$ generated by iterating the two functions

$$
\begin{aligned}
x_t &= h(s_t) \\
s_{t+1} &= g(s_t, x_t)
\end{aligned}
$$

starting from initial condition s_0 at $t = 0$, solves the original problem. A solution in the form of equations is said to be recursive.

To find the policy function h we need to know the value function $V(s)$, which expresses the optimal value of the original problem, starting from an arbitrary initial condition $s \in S$. Define

$$V(s_0) = \max_{\{x_s\}_{s=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(s_t, x_t)$$

subject to $s_{t+1} = g(s_t, x_t)$, with s_0 given.
We do not know $V(s_0)$ until after we have solved the problem, but if we knew it the policy function h could be computed by solving for each $s \in S$ the problem

$$\max_x \left\{ r(s, x) + \beta V(s') \right\}, \qquad \text{s.t. } s' = g(s, x) \tag{2}$$

Thus, we have exchanged the original problem of finding an infinite sequence of controls that maximizes expression (1) for the problem of finding the optimal value function $V(s)$ and a function h that solves the continuum of maximum problems (2) —one maximum problem for each value of s.

The function $V(s), h(s)$ are linked by the Bellman equation

$$V(s) = \max_x \{r(s,x) + \beta V[g(s,x)]\} \tag{3}$$

The maximizer of the RHS is a policy function $h(s)$ that satisfies

$$V(s) = r[s, h(s)] + \beta V\{g[s, h(s)]\} \tag{4}$$

This is a functional equation to be solved for the pair of unknown functions $V(s), h(s)$.

Some properties

Under various particular assumptions about r and g, it turns out that

1. The Bellman equation has a unique strictly concave solution.
2. This solution is approached in the limit as $j \to \infty$ by iterations on

$$V_{j+1}(s) = \max_x \{r(s,x) + \beta V_j(s')\}, \text{ s.t. } s' = g(s,x), s \text{ given}$$

   starting from any bounded and continuous initial V_0.
3. There is a unique and time-invariant optimal policy of the form $x_t = h(s_t)$, where h is chosen to maximize the RHS of the Bellman equation.
4. Off corners, the limiting value function V is differentiable.

Side note:

Banach Fixed-Point Theorem

Concave functions

- A real-valued function f on an interval (or, more generally, a convex set in vector space) is said to be concave if, for any x and y in the interval and for any $t \in [0, 1]$,

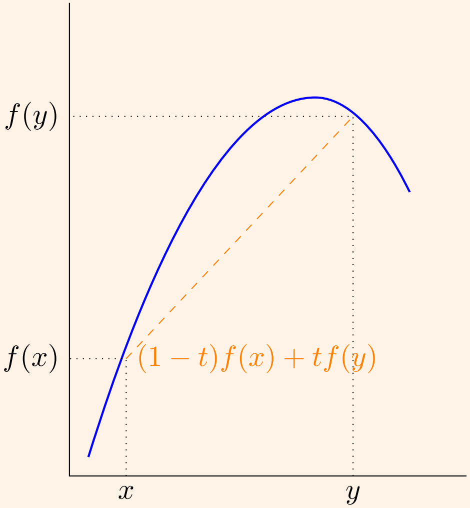
$$f((1-t)x + ty) \geq (1-t)f(x) + tf(y)$$

- A function is called strictly concave if

$$f((1-t)x + ty) \geq (1-t)f(x) + tf(y)$$

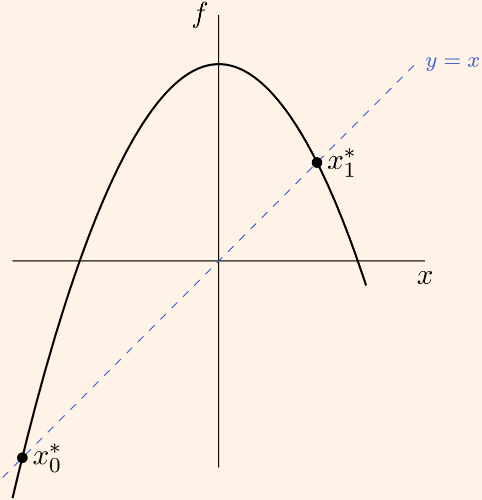
for any $t \in (0, 1)$ and $x \neq y$.

For a function $f : R \mapsto R$, this definition merely states that for every z between x and y, the point $(z, f(z))$ on the graph of f is above the straight line joining the points $(x, f(x))$ and $(y, f(y))$.



Fixed points

- A point x^* is a fixed-point of function f if it satisfies $f(x^*) = x^*$.
- Notice that $f(f(\dots f(x^*)\dots)) = x^*$.

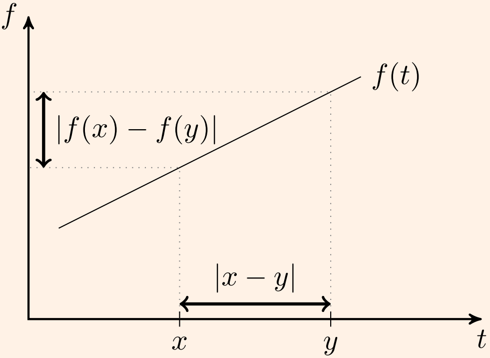


Contraction mappings

A mapping $f : X \mapsto X$ from a metric space X into itself is said to be a *strong contraction with modulus* δ, if $0 \leq \delta < 1$ and

$$d(f(x), f(y)) \leq \delta d(x, y)$$

for all x and y in X.



Banach Fixed-Point Theorem

If f is a strong contraction on a metric space X, then

- it possesses an unique fixed-point x^*, that is $f(x^*) = x^*$
- if $x_0 \in X$ and $x_{i+1} = f(x_i)$, then the x_i converge to x^*

Proof: Use x_0 and x^* in the definition of a strong contraction:

$$\begin{aligned} d(f(x_0), f(x_*)) &\leq \delta d(x_0, x^*) \Rightarrow \\ d(x_1, x_*) &\leq \delta d(x_0, x^*) \Rightarrow \\ d(x_k, x_*) &\leq \delta^k d(x_0, x^*) \to 0 \text{ as } k \to \infty \end{aligned}$$

Example 3:

Searching a fixed point by function iteration

- Consider finding a fixed point for the *function* $f(x) = 1 + 0.5x$, for $x \in \mathbb{R}$.
- It is easy to see that $x^* = 2$ is a fixed point:

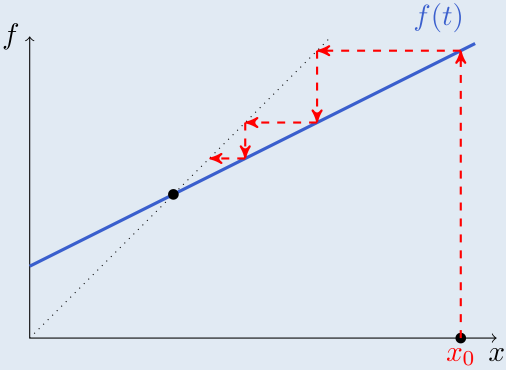
$$f(x^*) = f(2) = 1 + 0.5(2) = 2 = x^*$$

- Suppose we could not solve the equation $x = 1 + 0.5x$ directly. How could we find the fixed point then?
- Notice that $|f'(x)| = |0.5| < 1$, so f is a contraction.

By Banach Theorem, if we start from an arbitrary point x_0 and by iteration we form the sequence $x_{j+1} = f(x_j)$, it follows that $\lim_{j\to\infty} x_j = x^*$.

For example, pick:

$$x_0 = 6$$
$$x_1 = f(x_0) = 1 + \frac{6}{2} = 4$$
$$x_2 = f(x_1) = 1 + \frac{4}{2} = 3$$
$$x_3 = f(x_2) = 1 + \frac{3}{2} = 2.5$$
$$x_4 = f(x_3) = 1 + \frac{2.5}{2} = 2.25$$
$$\vdots$$



If we keep iterating, we will get arbitrarily close to the solution $x^* = 2$.

First-order necessary condition

Starting with the Bellman equation

$$V(s) = \max_x \{r(s,x) + \beta V[g(s,x)]\}$$

Since the value function is differentiable, the optimal $x^* \equiv h(s)$ must satisfy the first-order condition

$$r_x(s,x^*) + \beta V'\{g(s,x^*)\} g_x(s,x^*) = 0 \qquad \text{(FOC)}$$

Envelope condition

According to (4): $V(s) = r[s, h(s)] + \beta V\{g[s, h(s)]\}$
If we also assume that the policy function h(s) is differentiable, differentiation of this expression yields

$$\begin{aligned} V'(s) = {} & r_s[s, h(s)] + r_x[s, h(s)]h'(s) \\ & + \beta V'\{g[s, h(s)]\}\left\{g_s[s, h(s)] + g_x[s, h(s)]h'(s)\right\} \end{aligned}$$

Arranging terms, substituting $x^* = h(s)$ as the optimal policy

$$\begin{aligned} V'(s) = {} & r_s(s, x^*) + \beta V'[g(s, x^*)]g_s(s, x^*) \\ & + \left\{r_x[s, x^*] + \beta V'\{g[s, x^*]\}g_x[s, x^*]\right\} h'(s) \end{aligned}$$

Envelope condition (cont'n)

The highlighted part cancels out because of (FOC), therefore

$$V'(s) = r_s(s, x^*) + \beta V'\left(s'\right) g_s(s, x^*)$$

Notice that we could have obtained this result much faster by taking derivative of

$$V(s) = r(s, x^*) + \beta V[g(s, x^*)]$$

with respect to the state variable s as if the control variable $x^* \equiv h(s)$ did not depend on s.

Benveniste and Scheinkman formula

In the envelope condition

$$V'(s) = r_s(s, x^*) + \beta V'\left(s'\right) g_s(s, x^*)$$

when the states and controls can be defined in such a way that only x appears in the transition equation, i.e.,

$$s' = g(x) \qquad \Rightarrow \qquad g_s(s, x^*) = 0,$$

the derivative of the value function becomes

$$V'(s) = r_s[s, h(s)] \tag{B-S}$$

This is a version of a formula of Benveniste and Scheinkman.

Euler equations

- In many problems, there is no unique way of defining states and controls
- When the states and controls can be defined in such a way that $s' = g(x)$, the (FOC) for the Bellman equation together with the (B-S) formula implies

$$r_x(s_t, x_t) + \beta r_s(s_{t+1}, x_{t+1}) g'(x_t) = 0$$

- This equation is called an Euler equation.
- If we can write x_t as a function of s_{t+1}, we can use it to eliminate x_t from the Euler equation to produce a second-order difference equation in s_t.

Solving the Bellman equation

- In those cases in which we want to go beyond the Euler equation to obtain an explicit solution, we need to find the solution V of the Bellman equation (3)
- Given V, it is straightforward to solve (3) successively to compute the optimal policy.
- However, for infinite-horizon problems, we cannot use backward iteration.

Three computational methods

- There are three main types of computational methods for solving dynamic programs. All aim to solve the Bellman equation
  - Guess and verify
  - Value function iteration
  - Policy function iteration
- Each method is easier said than done: it is typically impossible analytically to compute even *one* iteration.
- Usually we need computational methods for approximating solutions: pencil and paper are insufficient.

Example 4:

Computer solution of DP models

There are several computer programs available for solving dynamic programming models:

- The CompEcon toolbox, a MATLAB toolbox accompanying Miranda and Fackler (2002) textbook.
- The PyCompEcon toolbox, my (still incomplete) Python version of Miranda and Fackler toolbox.
- Additional examples are available at quant-econ, a website by Sargent and Stachurski with Python and Julia scripts.

Guess and verify

- This method involves guessing and verifying a solution V to the Bellman equation.
- It relies on the uniqueness of the solution to the equation
- because it relies on luck in making a good guess, it is not generally available.

Value function iteration

- This method proceeds by constructing a sequence of value functions and associated policy functions.
- The sequence is created by iterating on the following equation, starting from $V_0 = 0$, and continuing until V_j has converged:

$$V_{j+1}(s) = \max_x \{r(s, x) + \beta V_j[g(s, x)]\}$$

Policy function iteration

This method, also known as *Howard's improvement algorithm*, consists of the following steps:

1. Pick a feasible policy, $x = h_0(s)$, and compute the value associated with operating forever with that policy:

$$V_{h_j}(s) = \sum_{t=0}^{\infty} \beta^t r[s_t, h_j(s_t)]$$

   where $s_{t+1} = g[s_t, h_j(s_t)]$, with $j = 0$.

2. Generate a new policy $x = h_{j+1}(s)$ that solves the two-period problem

$$\max_x \{r(s, x) + \beta V_{h_j}[g(s, x)]\}$$

   for each s.

3. Iterate over j to convergence on steps 1 and 2.

Stochastic control problems

Stochastic control problems

- We modify the transition equation and consider the problem of maximizing

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t r(s_t, x_t) \qquad \text{s.t. } s_{t+1} = g(s_t, x_t, \epsilon_{t+1}) \tag{5}$$

  with s_0 given at $t = 0$
- ϵ_t is a sequence of i.i.d. r.v. : $\mathbb{P}[\epsilon_t \leq e] = F(e)$ for all t
- ϵ_{t+1} is realized at $t+1$, after x_t has been chosen at t.
- At time t:
  - s_t is known
  - s_{t+j} is unknown $(j \geq 1)$

- The problem is to choose a *policy* or contingency plan $x_t = h(s_t)$. The Bellman equation is

$$V(s) = \max_x \left\{ r(s,x) + \beta\, \mathbb{E}[V\left(s'\right) \mid s] \right\}$$

- where $s' = g(s, x, \epsilon)$,
- and $E\{V\left(s'\right)|s\} = \int V\left(s'\right) dF(\epsilon)$
- The solution $V(s)$ of the B.E. can be computed by **value function iteration**.

- The FOC for the problem is

$$r_x(s,x) + \beta\,\mathbb{E}\left\{V'\left(s'\right) g_x(s,x,\epsilon) \mid s\right\} = 0$$

- When the states and controls can be defined in such a way that s does not appear in the transition equation,

$$V'(s) = r_s[s, h(s)]$$

- Substituting this formula into the FOC gives the stochastic Euler equation

$$r_x(s,x) + \beta\,\mathbb{E}\left\{r_s(s',x')g_x(s,x,\epsilon) \mid s\right\} = 0$$

Consumption and financial assets: infinite horizon

Consumption and financial assets

To ilustrate how dynamic programming works, we consider a intertemporal consumption problem.

The consumer

- Planning horizon: infinite
- Instant utility depends on current consumption: $u(c_t)$
- Constant utility discount rate $\beta \in (0, 1)$
- Lifetime utility is:

$$U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- The problem: choosing the optimal sequence of values $\{c_t^*\}$ that will maximize U, subject to a budget constraint.

A savings model

The consumer

- is endowed with A_0 units of the consumption good,
- does not have income
- can save in a bank deposit, which yields a interest rate r.

The budget constraint is

$$A_{t+1} = R(A_t - c_t)$$

where $R \equiv 1 + r$ is the *gross* interest rate.

The value function

- Once he chooses the sequence $\{c_t^*\}_{t=0}^{\infty}$ of optimal consumption, the maximum utility that he can achieved is ultimately constraint only by his initial assets A_0.
- So define the value function V as the maximum utility the consumer can get as a function of his initial assets

$$V(A_0) = \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $A_{t+1} = R(A_t - c_t)$

The consumer problem

Consumer problem:

$$V(A_0) = \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{(objective)}$$

$$A_{t+1} = R(A_t - c_t) \qquad \forall t = 0, 1, 2, \dots \qquad \text{(budget constraint)}$$

Dealing with the intertemporal budget constraint

Notice that we have a budget constraint for every time period t. So we form the Lagrangean

$$
\begin{aligned}
V(A_0) &= \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [R(A_t - c_t) - A_{t+1}] \\
&= \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left\{ \beta^t u(c_t) + \lambda_t [R(A_t - c_t) - A_{t+1}] \right\}
\end{aligned}
$$

Instead of dealing with the constraints explicitly, we can just substitute $c_t = A_t - A_{t+1}/R$ in all time periods:

$$
= \max_{\{A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left\{ \beta^t u \left(\underbrace{A_t - \frac{A_{t+1}}{R}}_{c_t} \right) \right\}
$$

So, we choose consumption implicitly by choosing the path of assets.

A recursive approach to solving the problem

Keeping in mind that $c_t = A_t - A_{t+1}/R$

$$
\begin{aligned}
V(A_0) &= \max_{\{A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&= \max_{\{A_{t+1}\}_{t=0}^{\infty}} \left\{ u(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_t) \right\} \\
&= \max_{\{A_{t+1}\}_{t=0}^{\infty}} \left\{ u(c_0) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right\}
\end{aligned}
$$

"An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

$$
\begin{aligned}
&= \max_{A_1} \left\{ u(c_0) + \beta \max_{\{A_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{t+1}) \right\} \\
&= \max_{A_1} \left\{ u(c_0) + \beta V(A_1) \right\}
\end{aligned}
$$

The Bellman equation

Bellman equation

$$V(A) = \max_{c,\ A'} \left\{ u(c) + \beta V(A') + \lambda[R(A-c) - A'] \right\}$$

- This says that the maximum lifetime utility the consumer can get must be equal to the sum of current utility plus the discounted value of the lifetime utility he will get starting next period.

Three ways to write the Bellman equation

- Explicitly writing down the budget constraint (BC):

$$V(A) = \max_{c,\ A'} \left\{u(c) + \beta V(A') + \lambda[R(A - c) - A']\right\}$$

- Using the BC to substitute future assets:

$$V(A) = \max_{c} \left\{u(c) + \beta V[R(A - c)]\right\}$$

- Using the BC to substitute consumption:

$$V(A) = \max_{A'} \left\{u\left(A - \frac{A'}{R}\right) + \beta V(A')\right\}$$

Obtaining the Euler equation

The problem is

$$V(A) = \max_{A'} \left\{ u\left(A - \frac{A'}{R}\right) + \beta V(A') \right\}$$

so the FOCs is

$$-\frac{u'(c)}{R} + \beta V'(A') = 0 \quad \Rightarrow u'(c) = \beta R V'(A')$$

The envelope condition is:

$$V'(A) = u'(c), \text{which implies that } V'(A') = u'(c')$$

Substituting into the FOC we get the:

Euler equation

$$u'(c) = \beta R u'(c')$$

Obtaining the Euler equation, second way

The Lagrangian for this problem is

$$V(A) = \max_{c,\ A'} \left\{ u(c) + \beta V(A') + \lambda[R(A-c) - A'] \right\}$$

so the FOCs are

$$\left.\begin{aligned} u'(c) &= \lambda R \\ \beta V'(A') &= \lambda \end{aligned}\right\} \Rightarrow u'(c) = \beta R V'(A')$$

and the envelope condition is

$$V'(A) = \lambda R = u'(c)$$

which implies that

$$V'(A') = u'(c') \qquad \Rightarrow \qquad u'(c) = \beta R u'(c')$$

(Not quite) obtaining the Euler equation

The problem is

$$V(A) = \max_{c} \{u(c) + \beta V[R(A - c)]\}$$

so the FOCs is

$$u'(c) - \beta R V'(A') = 0$$

but in this case the envelope condition is not useful:

$$V'(A) = \beta R V'(A')$$

The Euler equation

Euler equation

$$u'(c) = \beta R u'(c')$$

- This says that at the optimum, if the consumer gets one more unit of the good, he must be indifferent between consuming it now (getting $u'(c)$) or saving it (which increases next-period assets by R) an consuming it later, getting a discounted value of $\beta R u'(c')$.
- Notice that this is the say result we found on Lecture 8 (Applications of consumer theory), in the two-period intertemporal consumption problem!

Solving the Euler equation

Notice that the Euler equation can be written

$$u'\left(A_t - \frac{A_{t+1}}{R}\right) = \beta R u'\left(A_{t+1} - \frac{A_{t+2}}{R}\right)$$

which is a second-order nonlinear difference equation. In principle, it can be solved to obtain the

Policy function

$c_t^* = h(A_t)$	consumption function
$A_{t+1} = R[A_t - h(A_t)]$	asset accumulation

Consumption and financial assets: finite horizon

The consumer

- Planning horizon: T (possibly infinite)
- Instant utility depends on current consumption: $u(c_t) = \ln c_t$
- Constant utility discount rate $\beta \in (0, 1)$
- Lifetime utility is:

$$U(c_0, c_1, \ldots, c_T) = \sum_{t=0}^{T} \beta^t \ln c_t$$

- The problem: choosing the optimal values c_t^* that will maximize U, subject to a budget constraint.

A savings model

In this first model, the consumer

- is endowed with A_0 units of the consumption good,
- does not have income
- can save in a bank deposit, which yields a interest rate r.

The budget constraint is

$$A_{t+1} = R(A_t - c_t)$$

where $R \equiv 1 + r$ is the *gross* interest rate.

The value function

- Once he chooses the sequence $\{c_t^*\}_{t=0}^T$ of optimal consumption, the maximum utility that he can achieved is ultimately constraint only by his initial assets A_0 and by how many periods he lives $T+1$.
- So define the value function V as the maximum utility the consumer can get as a function of his initial assets

$$V_0(A_0) = \max_{\{c_t\}} \sum_{t=0}^{T} \beta^t \ln c_t^*$$

subject to $A_{t+1} = R(A_t - c_t)$

The consumer problem

Consumer problem:

$$V_0(A_0) = \max_{\{c,A\}} \sum_{t=0}^{T} \beta^t \ln c_t \qquad \text{(objective)}$$

$$A_{t+1} = R(A_t - c_t) \qquad \forall t = 0, \dots, T \quad \text{(budget constraint)}$$

$$A_{T+1} \geq 0 \qquad \text{(leave no debts)}$$

We now solve the problem for special cases $t = T$, $t = T - 1$, $t = T - 2$. Then we generalize for $T = \infty$.

Solution when $t = T$

In this case, consumer problem is simply

$$V_T(A_T) = \max_{c_T, A_{T+1}} \{\ln c_T\} \text{ subject to}$$
$$A_{T+1} = R(A_T - c_T), \qquad A_{T+1} \geq 0$$

We need to find c_T and A_{T+1}. Substitute $c_T = A_T - \frac{A_{T+1}}{R}$ in the objective function:

$$\max_{A_{T+1}} \ln\left[A_T - \frac{A_{T+1}}{R}\right] \quad \text{subject to} \quad A_{T+1} \geq 0$$

This function is strictly decreasing on A_{T+1}, so we set A_{T+1} to its minimum possible value; given the transversality constraint we set $A_{T+1} = 0$, which implies $c_T = A_T$ and $V_T(A_T) = \ln A_T$. In words, in his last period a consumer spends his entire assets.

Solution when $t = T - 1$

The problem is now

$$V_{T-1}(A_{T-1}) = \{\ln c_{T+1} + \beta \ln c_T\} \text{ subject to}$$
$$A_T = R\left(A_{T-1} - c_{T-1}\right),$$
$$A_{T+1} = R(A_T - c_T), \qquad A_{T+1} \geq 0$$

- We now need to find $c_{T-1},\ c_T,\ A_T$ and A_{T+1}.
- Instead of solving today for all these quantities, we proceed in two steps:
  - today (that is, in $T-1$) we solve only for c_{T-1} and A_T
  - and next period solving for the remaining c_T and A_{T+1}.
- But from the $t = T$ example we learned that a consumer will spend his entire assets in the last period, so $c_T = A_T$ (his remaining assets, which he will choose in the *current* period) and $A_{T+1} = 0$.

So we can rewrite the problem as

$$\begin{aligned} V_{T-1}(A_{T-1}) &= \max_{c_{T-1:T}, A_{T:T+1}} \{\ln c_{T-1} + \beta \ln c_T\} \\ &= \max_{c_{T-1}, A_T} \left\{\ln c_{T-1} + \beta \max_{c_T, A_{T+1}} [\ln c_T]\right\} \\ &= \max_{c_{T-1}, A_T} \{\ln c_{T-1} + \beta V_T(A_T)\} \end{aligned}$$

subject to $A_T = R(A_{T-1} - c_T)$

Again, we substitute $c_{T-1} = A_{T-1} - \frac{A_T}{R}$ and solve the problem

$$\max_{A_T} \left\{ \ln \left[A_{T-1} - \frac{A_T}{R} \right] + \beta V_T(A_T) \right\}$$

The first order condition is

$$\frac{1}{c_{T-1}}\frac{-1}{R} + \beta V_T'(A_T) = 0 \Rightarrow 1 = R\beta c_{T-1}V_T'(A_T)$$

Since $V_T(A) = \ln A$ (from the $t = T$ example), then $V_T'(A_T) = \frac{1}{A_T}$.
Substitute in the FOC

$$1 = R\beta c_{T-1}\tfrac{1}{A_T} \Rightarrow A_T^* = R\beta c_{T-1}^*$$

Now substitute in the BC to get $R\beta c_{T-1}^* = R(A_{T-1} - c_{T-1}^*)$. It follows that

$$c_{T-1}^* = \tfrac{1}{1+\beta}A_{T-1} \Rightarrow A_T^* = \tfrac{R\beta}{1+\beta}A_{T-1}$$

The value function is

$$
\begin{aligned}
V_{T-1}(A_{T-1}) &= \ln c^*_{T-1} + \beta V_T(A^*_T) \\
&= \ln c^*_{T-1} + \beta \ln A^*_T \\
&= \ln c^*_{T-1} + \beta \ln[R\beta c^*_{T-1}] \\
&= (1+\beta)\ln c^*_{T-1} + \beta \ln \beta + \beta \ln R \\
&= \begin{aligned} &(1+\beta)\ln A_{T-1} - (1+\beta)\ln(1+\beta) + \dots \\ &\qquad\qquad\qquad\qquad \dots + \beta\ln\beta + \beta\ln R \end{aligned} \\
&= (1+\beta)\ln A_{T-1} + \theta_{T-1}
\end{aligned}
$$

where the term θ_{T-1} is just a constant.

Solution when $t = T - 2$

The problem is now

$$V_{T-2}(A_{T-2}) = \max\left\{\ln c_{T-2} + \beta \ln c_{T-1} + \beta^2 \ln c_T\right\}$$

subject to

$$\begin{aligned} A_{T-1} &= R(A_{T-2} - c_{T-2}), \\ A_T &= R(A_{T-1} - c_{T-1}), \\ A_{T+1} &= R(A_T - c_T), \\ A_{T+1} &\geq 0 \end{aligned}$$

We will follow the same strategy as before: choose only c_{T-2} and A_{T-1} this period, and leave $c_{T-1}, c_T, A_T, A_{T+1}$ for next period.

$$\begin{aligned}
V_{T-2}(A_{T-2}) &= \max_{\substack{c_{T-2:T},\\ A_{T-1:T+1}}} \left\{\ln c_{T-2} + \beta \ln c_{T-1} + \beta^2 \ln c_T\right\} \\
&= \max_{c_{T-2}, A_{T-1}} \left\{ \ln c_{T-2} + \beta \max_{\substack{c_{T-1:T},\\ A_{T:T+1}}} \left[\ln c_{T-1} + \beta \ln c_T\right] \right\} \\
&= \max_{c_{T-2}, A_{T-1}} \left\{\ln c_{T-2} + \beta V_{T-1}(A_{T-1})\right\}
\end{aligned}$$

Again, we substitute $c_{T-2} = A_{T-2} - \frac{A_{T-1}}{R}$ and solve the problem

$$\max_{A_{T-1}} \left\{ \ln \left[A_{T-2} - \frac{A_{T-1}}{R} \right] + \beta V_{T-1}(A_{T-1}) \right\}$$

The first order condition is now

$$\frac{1}{c_{T-2}} \frac{-1}{R} + \beta V'_{T-1}(A_{T-1}) = 0 \Rightarrow 1 = R\beta c_{T-2} V'_{T-1}(A_{T-1})$$

But $V_{T-1}(A) = (1+\beta)\ln A + \theta_{T-1}$ (from the $t = T-1$ step). Therefore $V'_{T-1}(A_{T-1}) = \frac{1+\beta}{A_{T-1}}$. Substitute in the FOC

$$1 = R\beta c_{T-2} \tfrac{1+\beta}{A_{T-1}} \qquad \Rightarrow \quad A^*_{T-1} = R(\beta + \beta^2) c^*_{T-2}$$

Now substitute in the budget constraint to get $(1+\beta)R\beta c^*_{T-2} = R(A_{T-2} - c^*_{T-2})$. Then

$$c^*_{T-2} = \tfrac{1}{1+\beta+\beta^2} A_{T-2} \quad \Rightarrow \quad A^*_{T-1} = \tfrac{R(\beta+\beta^2)}{1+\beta+\beta^2} A_{T-2}$$

and the value function is

$$\begin{aligned}
V_{T-2}(A_{T-2}) &= \ln c^*_{T-2} + \beta V_{T-1}(A^*_{T-1}) \\
&= \ln c^*_{T-2} + \beta[(1+\beta)\ln(A^*_{T-1}) + \theta_{T-1}] \\
&= \ln c^*_{T-2} + (\beta+\beta^2)\ln[R(\beta+\beta^2)c^*_{T-2}] + \beta\theta_{T-1} \\
&= \begin{array}{r}(1+\beta+\beta^2)\ln c^*_{T-2} + (\beta+\beta^2)[\ln R + \dots \\ \dots + \ln(\beta+\beta^2)] + \beta\theta_{T-1}\end{array} \\
&= (1+\beta+\beta^2)\ln A_{T-2} + \theta_{T-2}
\end{aligned}$$

where
$\theta_{T-2} = (\beta+2\beta^2)\ln R + (\beta+2\beta^2)\ln\beta - (1+\beta+\beta^2)\ln(1+\beta+\beta^2)$

Solution when $t = T - K$

If we keep iterating, for $t = T - K$ the problem would be

$$V_{T-K}(A_{T-K}) = \max\left\{\ln c_{T-K} + \beta \ln c_{T-K+1} + \cdots + \beta^K \ln c_T\right\}$$

subject to

$$\begin{aligned} A_{t+1} &= R(A_t - c_t), \text{ for } t = T - K, T - K + 1, \ldots, T \\ A_{T+1} &\geq 0 \end{aligned}$$

We will follow the same strategy as before: choose only c_{T-K} and A_{T-K+1} this period, and leave the other variables for next period.

$$V_{T-K}(A_{T-K}) =$$

$$\max_{\substack{c_{T-K},\\ A_{T-K+1}}} \left\{ \ln c_{T-K} + \beta \max_{\substack{c_{T-K+1:T},\\ A_{T-K+2:T+1}}} \left[\ln c_{T-K+1} + \cdots + \beta^{K-1} \ln c_T\right] \right\}$$

$$= \max_{\substack{c_{T-K},\\ A_{T-K+1}}} \left\{ \ln c_{T-K} + \beta V_{T-K+1}(A_{T-K+1}) \right\}$$

Again, we substitute $c_{T-K} = A_{T-K} - \frac{A_{T-K+1}}{R}$ and solve the problem

$$\max_{A_{T-K}} \left\{ \ln\left[A_{T-K} - \frac{A_{T-K+1}}{R}\right] + \beta V_{T-K+1}(A_{T-K+1}) \right\}$$

The first order condition is now

$$\frac{1}{c_{T-K}}\frac{-1}{R} + \beta V'_{T-K+1}(A_{T-K+1}) = 0$$

which can be written as

$$1 = R\beta c_{T-K} V'_{T-K+1}(A_{T-K+1})$$

But now we don't know $V_{T-K+1}(A)$, unless we solve for all intermediate steps. Instead of doing that, we will search for patterns in our results.

Searching for patterns

Let's summarize the results for the policy function.

t	c_t^*	A_{t+1}^*
T	A_T	$0A_T$
$T-1$	$\frac{1}{1+\beta}A_{T-1}$	$R\beta\frac{1}{1+\beta}A_{T-1}$
$T-2$	$\frac{1}{1+\beta+\beta^2}A_{T-2}$	$R\beta\frac{1+\beta}{1+\beta+\beta^2}A_{T-2}$
We could guess that after K iterations:		
$T-K$	$\frac{1}{1+\beta+\dots+\beta^K}A_{T-K}$	$R\beta\frac{1+\beta+\dots+\beta^{K-1}}{1+\beta+\dots+\beta^K}A_{T-K}$
$=$	$\dfrac{1-\beta}{1-\beta^{K+1}}A_{T-K}$	$R\beta\dfrac{1-\beta^K}{1-\beta^{K+1}}A_{T-K}$

The time path of assets

Since $A_{T-K+1} = R\beta\frac{1-\beta^K}{1-\beta^{K+1}}A_{T-K}$, setting $K = T, T-1$:

$$\begin{aligned}
A_1 &= R\beta\frac{1-\beta^T}{1-\beta^{T+1}}A_0 \\
A_2 &= R\beta\frac{1-\beta^{T-1}}{1-\beta^T}A_1 \\
&= (R\beta)^2\frac{1-\beta^{T-1}}{1-\beta^{T+1}}A_0
\end{aligned}$$

Iterating in this fashion we find that

$$A_t = (R\beta)^t\frac{1-\beta^{T+1-t}}{1-\beta^{T+1}}A_0$$

The time path of consumption

Since $c^*_{T-K} = \frac{1-\beta}{1-\beta^{K+1}} A_{T-K}$, setting $t = T - K$:
Then consumption

$$\begin{aligned} c^*_t &= \frac{1-\beta}{1-\beta^{T+1-t}} A_t \\ &= \frac{1-\beta}{1-\beta^{T+1-t}} \left[(R\beta)^t \frac{1-\beta^{T+1-t}}{1-\beta^{T+1}} A_0 \right] \\ &= (R\beta)^t \underbrace{\frac{1-\beta}{1-\beta^{T+1}} A_0}_{\phi} \end{aligned}$$

That is

$$\ln c^*_t = t \ln(R\beta) + \ln \phi$$

The time 0 value function

Substitution of the optimal consumption path in the Bellman equation give the value function

$$
\begin{aligned}
V_0(A_0) &\equiv \sum_{t=0}^{T} \beta^t \ln c_t^* = \sum_{t=0}^{T} \beta^t \left(t \ln(R\beta) + \ln\phi\right) \\
&= \ln(R\beta) \sum_{t=0}^{T} \beta^t t + \ln\phi \sum_{t=0}^{T} \beta^t \\
&= \frac{\beta}{1-\beta}\left(\frac{1-\beta^T}{1-\beta} - T\beta^T\right)\ln(R\beta) + \frac{1-\beta^{T+1}}{1-\beta}\ln\phi \\
&= \begin{array}{l} \frac{\beta}{1-\beta}\left(\frac{1-\beta^T}{1-\beta} - T\beta^T\right)\ln(R\beta) + \dots \\ \quad + \frac{1-\beta^{T+1}}{1-\beta}\ln\frac{1-\beta}{1-\beta^{T+1}} + \frac{1-\beta^{T+1}}{1-\beta}\ln A_0 \end{array}
\end{aligned}
$$

From finite horizon to infinite horizon

Our results so far are

$$A_t = (R\beta)^t \frac{1-\beta^{T+1-t}}{1-\beta^{T+1}} A_0 \qquad c_t^* = (R\beta)^t \frac{1-\beta}{1-\beta^{T+1}} A_0$$

$$V_0(A_0) = \frac{\beta}{1-\beta}\left(\frac{1-\beta^T}{1-\beta} - T\beta^T\right)\ln(R\beta) + \frac{1-\beta^{T+1}}{1-\beta}\ln\frac{1-\beta}{1-\beta^{T+1}} + \frac{1-\beta^{T+1}}{1-\beta}\ln A_0$$

Taking the limit as $T \to \infty$

$$A_t = (R\beta)^t A_0 \qquad c_t^* = (R\beta)^t (1-\beta) A_0 = (1-\beta)A_t$$

$$V_0(A_0) = \frac{1}{1-\beta}\ln A_0 + \frac{\beta\ln R + \beta\ln\beta + (1-\beta)\ln(1-\beta)}{(1-\beta)^2}$$

The policy function

Policy function

$$
\begin{aligned}
c_t^* &= (1-\beta)A_t && \text{consumption function} \\
A_{t+1} &= R\beta A_t && \text{asset accumulation}
\end{aligned}
$$

- This says that the optimal consumption rule is, in every period, to consume a fraction $1-\beta$ of available initial assets.
- Over time, assets will increase, decrease or remain constant depending on how the degree of impatience β compares to reward to postpone consumption R.

Time-variant value function

Now let's summarize the results for the value function:

t	$V_t(A)$
T	$\ln A$
$T-1$	$(1+\beta)\ln A + \theta_{T-1}$
$T-2$	$(1+\beta+\beta^2)\ln A + \theta_{T-2}$
	$\vdots$
0	$\frac{1}{1-\beta}\ln A + \frac{\beta}{1-\beta}\left(\frac{1-\beta^T}{1-\beta} - T\beta^T\right)\ln(R\beta) + \frac{1}{1-\beta}\ln\frac{1-\beta}{1-\beta^{T+1}}$

Notice that the value function changes each period, but only because each period the remaining horizon becomes one period shorter.

Time-invariant value function

Remember that in our K iteration,

$$V_{T-K}(A_{T-K}) = \max_{\substack{c_{T-K},\\ A_{T-K+1}}} \{\ln c_{T-K} + \beta V_{T-K+1}(A_{T-K+1})\}$$

With an infinite horizon, the remaining horizon is the same in $T-K$ and in $T-K+1$, so the value function is the same, precisely the fixed-point of the Bellman equation. Then we can write

$$V(A_{T-K}) = \max_{\substack{c_{T-K},\\ A_{T-K+1}}} \{\ln c_{T-K} + \beta V(A_{T-K+1})\}$$

or simply

$$V(A) = \max_{c,A'} \left\{\ln c + \beta V(A')\right\}$$

where a prime indicates a next-period variable

The first order condition

Using the budget constraint to substitute consumption

$$V(A) = \max_{A'} \left\{ \ln\left(A - \frac{A'}{R}\right) + \beta V(A') \right\}$$

we obtain the FOC:

$$1 = R\beta c V'(A')$$

Despite not knowing V, we can determine its first derivative using the envelope condition.Thus, from

$$V(A) = \ln\left(A - \frac{A'^*}{R}\right) + \beta V(A'^*)$$

we get

$$V'(A) = \frac{1}{c}$$

The Euler condition

- Because the solution is time-invariant, $V'(A) = \frac{1}{c}$ implies that $V'(A') = \frac{1}{c'}$.
- Substitute this into the FOC to obtain the

Euler equation

$$1 = R\beta \frac{c}{c'} = R\beta \frac{u'(c')}{u'(c)}$$

- This says that the marginal rate of substitution of consumption between any consecutive periods $\frac{u'(c)}{\beta u'(c')}$ must equal the relative price of the later consumption in terms of the earlier consumption R.

Value function iteration

- Suppose we wanted to solve the infinite horizon problem

$$V(A) = \max_{c,A'} \left\{\ln c + \beta V(A')\right\} \text{ subject to} A' = R(A-c)$$

by value function iteration:

$$V_{j+1}(A) = \max_{c,A'} \left\{\ln c + \beta V_j(A')\right\} \text{ subject to} A' = R(A-c)$$

- If we start iterating from $V_0(A) = 0$, our iterations would look identical to the procedure we used to solve for the finite horizon problem!

- Then, our iterations would look like

j	$V_j(A)$
0	0
1	$\ln A$
2	$(1+\beta)\ln A + \theta_2$
3	$(1+\beta+\beta^2)\ln A + \theta_3$
	$\vdots$

- If we keep iterating, we would expect that the coefficient on $\ln A$ would converge to $1+\beta+\beta^2+\cdots = \frac{1}{1-\beta}$
- However, it is much harder to see a pattern on the θ_j sequence.
- Then, we could try now the guess and verify, guessing that the solution takes the form $V(A) = \frac{1}{1-\beta}\ln A + \theta$.

- Our guess: $V(A) = \frac{1}{1-\beta} \ln A + \theta$
- Solution must satisfy the FOC: $1 = R\beta c V'(A')$ and budget constraint $A' = R(A - c)$.
- Combining these conditions we find $c^* = (1-\beta)A$ and $A'^* = R\beta A$.
- To be a solution of the Bellman equation, it must be the case that both sides are equal:

LHS	RHS
$V(A)$	$\ln c^* + \beta V(A'^*)$
$\frac{1}{1-\beta}\ln A + \theta$	$= \ln(1-\beta)A + \beta\left[\frac{\ln A'^*}{1-\beta} + \theta\right]$ $= \ln(1-\beta)A + \beta\left[\frac{\ln R\beta A}{1-\beta} + \theta\right]$ $= \frac{1}{1-\beta}\ln A + \frac{\beta}{1-\beta}\ln R\beta + \ln(1-\beta) + \beta\theta$

The two sides are equal if and only if

$$\theta = \frac{\beta}{1-\beta}\ln R\beta + \ln(1-\beta) + \beta\theta$$

That is, if

$$\theta = \frac{\beta\ln R + \beta\ln\beta + (1-\beta)\ln(1-\beta)}{(1-\beta)^2}$$

Why the envelope condition works?

The last point in our discussion is to justify the envelope condition: deriving $V(A)$ pretending that A'^* did not depend on A. But we know it does, so write $A'^* = h(A)$ for some function h. From the definition of the value function write:

$$V(A) = \ln\left[A - \frac{h(A)}{R}\right] + \beta V(h(A))$$

Take derivative and arrange terms:

$$\begin{aligned} V'(A) &= \frac{1}{c}\left[1 - \frac{h'(A)}{R}\right] + \beta V'(h(A))h'(A) \\ &= \frac{1}{c} + \left[\frac{-1}{cR} + \beta V'(A'^*)\right] h'(A) \end{aligned}$$

but the term in square brackets must be zero from the FOC.

Consumption and physical investment

A model with production

In this model

- the consumer is endowed with k_0 units of a good that can be used either for consumption or for the production of additional good
- we refer to "capital" to the part of the good that is used for future production
- capital fully depreciates with the production process.
- The lifetime utility of the consumer is again $U(c_0, c_1, \dots, c_\infty) = \sum_{t=0}^{\infty} \beta^t \ln c_t$,
- The production function is $y = Ak^\alpha$, where $A > 0$ and $0 < \alpha < 1$ are parameters.
- The budget constraint is $c_t + k_{t+1} = Ak_t^\alpha$.

The consumer problem

Consumer problem:

$$V(k_0) = \max_{\{c,k'\}} \sum_{t=0}^{\infty} \beta^t \ln c_t \qquad \text{(objective)}$$

$$k' = Ak^\alpha - c \qquad \text{(resource constraint)}$$

The Bellman equation

- In this case, the Bellman equation is

$$V(k_0) = \max_{c_0, k_1} \{\ln c_0 + \beta V(k_1)\}$$

- Substitute the constraint $c_0 = Ak_0^\alpha - k_1$ in the BE. To simplify notation, we drop the time index and use a prime (as in k') to denote "next period" variables. Then, BE is

$$V(k) = \max_{k'} \left\{\ln(Ak^\alpha - k') + \beta V(k')\right\}$$

- We will solve this equation by value function iteration.

The Euler equation

Remember that $u(c) = \ln c$, $y = f(k) = Ak^{\alpha}$, so Bellman equation can be written as:

$$V(k) = \max_{k'} \left\{ u\left(f(k) - k'\right) + \beta V(k') \right\}$$

we get the FOC $u'(c) = \beta V'(k')$ and the envelope condition $V'(k) = u'(c) f'(k)$

Euler equation

$$u'(c) = \beta f'(k') u'(c')$$

Notice how this result is similar to the one we got in the savings model: the return for giving up one unit of current consumption is:

savings model: $R = 1 + r$, the gross interest rate.

physical capital model: $f'(k')$, the marginal product of capital.

Solving Bellman equation by function iteration

- How do we solve the Bellman equation?

$$V(k) = \max_{k'} \left\{ \ln(Ak^\alpha - k') + \beta V(k') \right\}$$

- This equation involves a *functional*, where the unknown is the function $V(k)$.
- Unfortunately, we cannot solve for V directly.
- However, this equation is a contraction mapping (as long as $|\beta| < 1$) that has a fixed point (its solution).
- Let's pick an initial guess ($V_0(k) = 0$ is a convenient one) and them iterate over the Bellman equation by*

$$V_{j+1}(k) = \max_{k'} \left\{ \ln(Ak^\alpha - k') + \beta V_j(k') \right\}$$

*The j subscript refers to an iteration, not to the horizon.

Starting from $V_0 = 0$, the problem becomes:

$$V_1(k) = \max_{k'} \left\{ \ln(Ak^\alpha - k') + \beta \times 0 \right\}$$

Since the objective is decreasing on k' and we have the restriction $k' \geq 0$, the solution is simply $k'^* = 0$. Then $c^* = Ak^\alpha$

$$\begin{aligned} V_1(k) &= \ln c^* + \beta \times 0 \\ &= \ln A + \alpha \ln k \end{aligned}$$

This completes our first iteration. Let's now find V_2:

$$V_2(k) = \max_{k'} \left\{ \ln(Ak^\alpha - k') + \beta[\ln A + \alpha \ln k'] \right\}$$

FOC is

$$\frac{1}{Ak^{\alpha} - k'} = \frac{\alpha\beta}{k'} \quad \Rightarrow \quad k'^{*} = \frac{\alpha\beta}{1+\alpha\beta} Ak^{\alpha} = \theta_1 Ak^{\alpha}$$

Then consumption is $c^{*} = (1-\theta_1)Ak^{\alpha} = \frac{1}{1+\alpha\beta}Ak^{\alpha}$ and

$$\begin{aligned} V_2(k) &= \ln(c^{*}) + \beta \ln A + \alpha\beta \ln k'^{*} \\ &= \ln(1-\theta_1) + \ln(Ak^{\alpha}) + \beta[\ln A + \alpha \ln \theta_1 + \alpha \ln(Ak^{\alpha})] \\ &= (1+\alpha\beta)\ln(Ak^{\alpha}) + \beta \ln A + [\ln(1-\theta_1) + \alpha\beta \ln \theta_1] \\ &= (1+\alpha\beta)\ln(Ak^{\alpha}) + \phi_1 \end{aligned}$$

This completes the second iteration.

Let's have one more:

$$V_3(k) = \max_{k'} \left\{ \ln(Ak^\alpha - k') + \beta[(1+\alpha\beta)\ln(Ak'^\alpha) + \phi_1] \right\}$$

The FOC is

$$\begin{aligned} \frac{1}{Ak^\alpha - k'} &= \frac{\alpha\beta(1+\alpha\beta)}{k'} \\ k'^* &= \frac{\alpha\beta + \alpha^2\beta^2}{1+\alpha\beta+\alpha^2\beta^2} Ak^\alpha = \theta_2 Ak^\alpha \end{aligned}$$

Then consumption is $c^* = (1-\theta_2)Ak^\alpha = \frac{1}{1+\alpha\beta+\alpha^2\beta^2} Ak^\alpha$

You might be tired by now of iterating this function. Me too! So let's try to find some patterns (unless you really want to iterate to infinity). Let's summarize the results for the consumption policy function.

j	c^*
1	$(1)^{-1}Ak^{\alpha}$
2	$(1+\alpha\beta)^{-1}Ak^{\alpha}$
3	$(1+\alpha\beta+\alpha^2\beta^2)^{-1}Ak^{\alpha}$

From this table, we could guess that after j iterations, the consumption policy would look like:

$$c_j^* = (1+\alpha\beta+\ldots+\alpha^j\beta^j)^{-1}Ak^{\alpha}$$

Iterating to infinity

- To converge to the fixed point, we need to iterate to infinity.
- Simply take the limit $j \to \infty$ of the consumption function: since $0 < \alpha\beta < 1$, the geometric series converges, and so

$$c^* = (1 - \alpha\beta)Ak^\alpha$$

$$k'^* = \alpha\beta Ak^\alpha$$

The time path of capital and consumption

Optimal capital evolves according to:

$$\begin{aligned}
\ln k_1^* &= \ln(\alpha\beta A) + \alpha \ln k_0 \\
&= (1-\alpha)\psi + \alpha \ln k_0 \Rightarrow \\
\ln k_1^* - \psi &= \alpha(\ln k_0 - \psi) \Rightarrow \\
\ln k_t^* - \psi &= \alpha^t(\ln k_0 - \psi) \Rightarrow \\
\ln k_t^* &= \psi(1-\alpha^t) + \alpha^t \ln k_0
\end{aligned}$$

$$\psi \equiv \frac{\ln(\alpha\beta A)}{1-\alpha}$$

Optimal consumption is then:

$$\begin{aligned}
\ln c_t^* &= \ln[A(1-\alpha\beta)] + \alpha \ln k_t^* \\
&= \ln[A(1-\alpha\beta)] + \alpha\psi(1-\alpha^t) + \alpha^{t+1} \ln k_0
\end{aligned}$$

The value function is then:

$$
\begin{aligned}
V(k_0) &\equiv \sum_{t=0}^{\infty} \beta^t \ln(c_t^*) \\
&= \sum_{t=0}^{\infty} \left\{\beta^t \ln[A(1-\alpha\beta)] + \alpha\psi\beta^t(1-\alpha^t) + \beta^t\alpha^{t+1}\ln k_0\right\} \\
&= \frac{\ln[A(1-\alpha\beta)]}{1-\beta} + \alpha\psi\left[\frac{1}{1-\beta} - \frac{1}{1-\alpha\beta}\right] + \frac{\alpha\ln k_0}{1-\alpha\beta} \\
&= \frac{\ln[A(1-\alpha\beta)]}{1-\beta} + \alpha\frac{\ln(\alpha\beta A)}{1-\alpha}\left[\frac{\beta(1-\alpha)}{(1-\beta)(1-\alpha\beta)}\right] + \frac{\alpha\ln k_0}{1-\alpha\beta} \\
&= \frac{\ln[A(1-\alpha\beta)]}{1-\beta} + \frac{\alpha\beta\ln(\alpha\beta A)}{(1-\beta)(1-\alpha\beta)} + \frac{\alpha\ln k_0}{1-\alpha\beta}
\end{aligned}
$$

Solving by guess and verify

- Since we already know the answer, we'll guess a function of the correct form, but leave its coefficients undetermined.
- This is called the method of undetermined coefficients.
- Thus, we make the guess $V(k) = E + F\ln k$ where E and F are undetermined constants.
- In this case, the Bellman equation is

$$V(k) = \max_{k'}\left\{\ln(Ak^\alpha - k') + \beta E + \beta F\ln k'\right\}$$

- FOC is

$$\frac{1}{Ak^\alpha - k'} = \frac{\beta F}{k'} \quad\Rightarrow\quad k'^* = \tfrac{\beta F}{1+\beta F}Ak^\alpha \quad\Rightarrow\quad c^* = \tfrac{1}{1+\beta F}Ak^\alpha$$

Substitute in the Bellman equation is

$$
\begin{aligned}
V(k) &= \ln c^* + \beta E + \beta F \ln k'^* \\
E + F \ln k &= \ln\left(\tfrac{1}{1+\beta F} A k^\alpha\right) + \beta E + \beta F \ln\left(\tfrac{\beta F}{1+\beta F} A k^\alpha\right) \\
&= \ln \tfrac{A}{1+\beta F} + \alpha \ln k + \beta E + \beta F \ln \tfrac{A\beta F}{1+\beta F} + \alpha\beta F \ln k \\
&= \left\{\ln \tfrac{A}{1+\beta F} + \beta E + \beta F \ln \tfrac{A\beta F}{1+\beta F}\right\} + \alpha(1+\beta F) \ln k
\end{aligned}
$$

Therefore

$$F = \alpha(1+\beta F) \quad \Rightarrow \quad F = \frac{\alpha}{1-\alpha\beta}$$

$1 + \beta F = \frac{1}{1-\alpha\beta}$

And

$\frac{\beta F}{1+\beta F} = \alpha\beta$

$\frac{A}{1+\beta F} = A(1-\alpha\beta)$

$$(1-\beta)E = \ln \frac{A}{1+\beta F} + \beta F \ln \frac{A\beta F}{1+\beta F}$$

$$E = \frac{1}{1-\beta}\left\{\ln[A(1-\alpha\beta)] + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta A)\right\}$$

Finally, substitute in FOC to get:

$$k'^* = \frac{\beta F}{1+\beta F} Ak^\alpha = \alpha\beta Ak^\alpha$$

$$c^* = \frac{1}{1+\beta F} Ak^\alpha = (1-\alpha\beta)Ak^\alpha$$

$$\begin{aligned} V(k) &= E + F\ln k \\ &= \frac{1}{1-\beta}\left\{\ln[A(1-\alpha\beta)] + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta A)\right\} + \frac{\alpha}{1-\alpha\beta}\ln k \end{aligned}$$

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