

Applications of consumer theory

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Table of contents

1. The Work-Leisure Decision
2. Choice under uncertainty
3. Intertemporal consumption
4. Intertemporal consumption with uncertainty

The Work-Leisure Decision

The setup

- ▶ There are only two goods: consumption goods C and time.
- ▶ Barter economy: consumer exchanges work time for consumption good.
 - ▶ Price of consumption is 1.
 - ▶ One hour of work is worth w units of consumption.
- ▶ Consumer is endowed with h hours, to be used in:
 - leisure: l = time used at home
 - work: N^s = time exchanged in the market (labor time)
- ▶ The time constraint for the consumer is then

$$l + N^s = h$$

which states that leisure time plus time spent working must sum to total time available.

The consumer's real disposable income

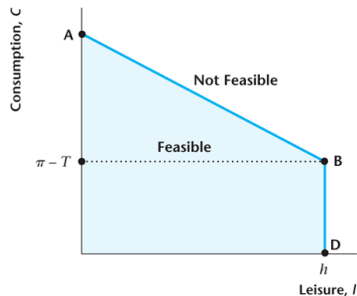
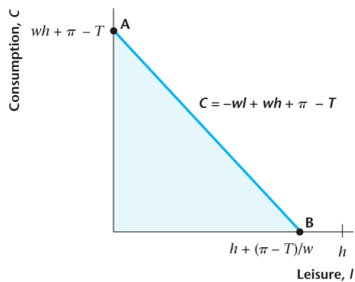
- ▶ For his work, consumer gets $wN^s = w(h - l)$ units of consumption good.
- ▶ Consumer also receives π units of consumption good, in the form of real dividend income.
- ▶ Consumer must pay a lump-sum tax amount T to the government.
- ▶ Therefore, the budget constraint is

$$C = w(h - l) + \pi - T$$

- ▶ which can also be written as

$$C + wl = wh + \pi - T$$

The budget constraint



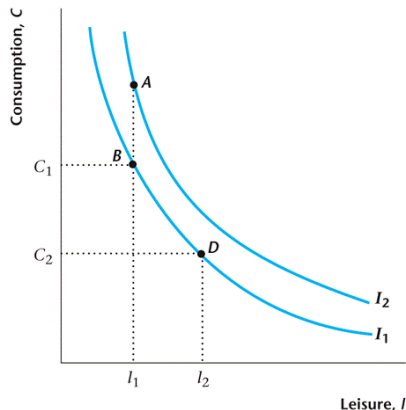
The consumer's preferences

The representative consumer's preferences are defined by

$$U(C, l)$$

with $U(\cdot, \cdot)$ a function that is:

- ▶ increasing in both arguments,
- ▶ strictly quasiconcave, and
- ▶ twice differentiable.



The consumer's problem

- ▶ The consumer's optimization problem is to choose C and l so as to maximize $U(C, l)$ subject to his or her budget constraint—that is,

$$\max_{C, l} U(C, l) \quad \text{s.t.} \quad \begin{cases} C = w(h - l) + \pi - T \\ l \leq h \end{cases}$$

- ▶ This problem is a constrained optimization problem, with the associated Lagrangian

$$\mathcal{L} = U(C, l) + \lambda[w(h - l) + \pi - T - C] + \mu(h - l)$$

where λ and μ are the Lagrange multipliers.

Solving the problem

- ▶ We assume that there is an interior solution to the consumer's problem where $C > 0$ and $0 < l$.
- ▶ This can be guaranteed by assuming that

$$U_C(0, l) = \infty \quad \text{and} \quad U_l(C, 0) = \infty$$

- ▶ The first-order conditions are

$$\begin{aligned}U_C(C, l) - \lambda &= 0 \\U_l(C, l) - \lambda w - \mu &= 0 \\w(h - l) + \pi - T - C &= 0.\end{aligned}$$

- ▶ Slackness conditions:

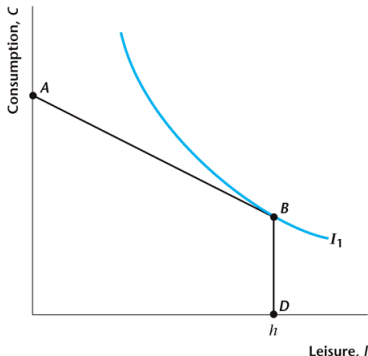
$$\mu \geq 0 \quad h - l \geq 0 \quad \mu(h - l) = 0$$

Case 1: $l = h$ (consumer does not work!)

- ▶ For this case to be feasible, we require that $C = \pi - T > 0$.
- ▶ From the first two FOCs and nonnegativity of multiplier:

$$U_l(\pi - T, h) - wU_C(\pi - T, h) = \mu \geq 0$$
$$\Leftrightarrow w \leq \frac{U_l(\pi - T, h)}{U_C(\pi - T, h)}$$

- ▶ Thus, consumer does not work if he has $\pi - T > 0$, and at bundle $(\pi - T, h)$ the market wage rate is less than his MRS of leisure for consumption.
- ▶ In a competitive equilibrium we cannot have $l = h$, as this would imply that nothing would be produced and $C = 0$.

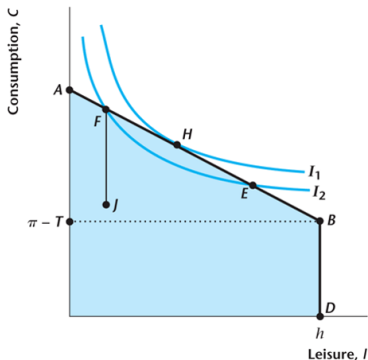


Case 2: $\mu = 0$ (consumer goes to work!)

- ▶ From the first two FOCs:

$$U_l(C^*, l^*) = wU_C(C^*, l^*)$$
$$\Leftrightarrow w = \frac{U_l(C^*, l^*)}{U_C(C^*, l^*)}$$

- ▶ Thus, consumer works $N^{S^*} = h - l^*$ hours and consumes $C^* = w(h - l^*) + \pi - T$.
- ▶ At this allocation, his MRS of leisure for consumption equals the market wage rate.



A parametric example

$$U(C, l) = \ln(c) + \gamma \ln(l)$$

- ▶ FOC

$$\text{MRS}_{lC} = \frac{U_l}{U_C} = \frac{\frac{\gamma}{l}}{\frac{1}{C}} = \frac{\gamma C}{l} = w$$

- ▶ Time and budget constraints:

$$w = \frac{\gamma C}{h - N^s}$$

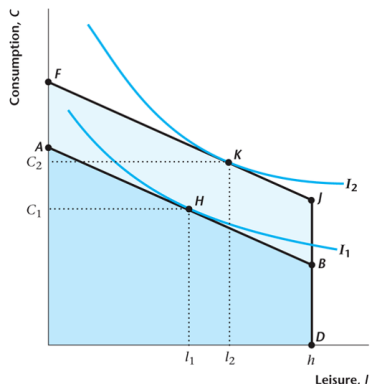
$$C = wN^s + \pi - T$$

- ▶ Then

$$N^{s*} = \frac{wh - \gamma(\pi - T)}{(1 + \gamma)w}$$

Real Dividends or Taxes Change for the Consumer

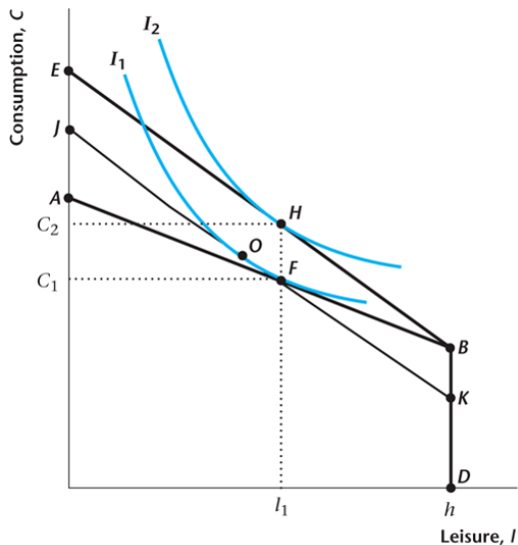
- ▶ Assume that consumption and leisure are both normal goods.
- ▶ An increase in dividends or a decrease in taxes will then cause the consumer to increase consumption and reduce the quantity of labor supplied (increase leisure).



An Increase in the Market Real Wage Rate

- ▶ This has income and substitution effects.
- ▶ **Substitution effect:** the price of leisure rises, so the consumer substitutes from leisure to consumption.
- ▶ **Income effect:** the consumer is effectively more wealthy and, since both goods are normal, consumption increases and leisure increases.
- ▶ **Conclusion:** Consumption must rise, but leisure may rise or fall.

Increase in the Real Wage Rate—Income and Substitution Effects



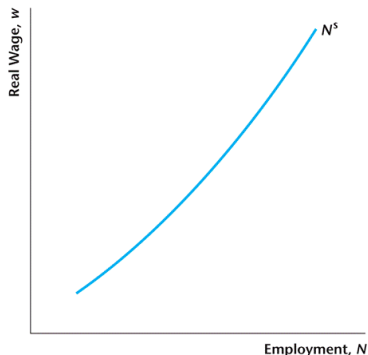
The labor supply function

- ▶ Suppose $l(w)$ is a function that tells us how much leisure the consumer wishes to consume, given the real wage w .
- ▶ Then, the labor supply curve is given by

$$N^s(w) = h - l(w)$$

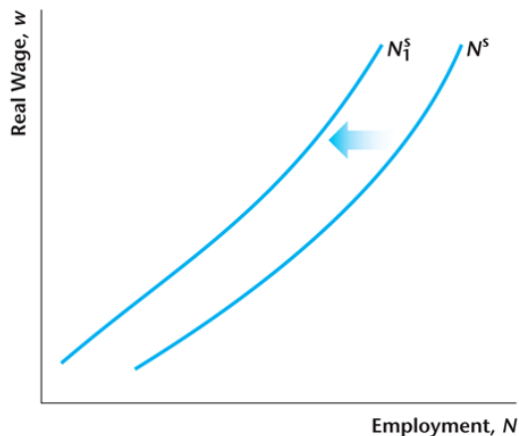
The slope of the labor supply function

- ▶ We do not know whether labor supply is increasing or decreasing in the real wage, because the effect of a wage increase on the consumer's leisure choice is ambiguous.
- ▶ Assuming that the substitution effect is larger than the income effect of a change in the real wage, labor supply increases with an increase in the real wage, and the labor supply schedule is upward-sloping.



Labor supply response to an increase in dividend

- ▶ An increase in nonwage disposable income shifts the labor supply curve to the left, that is, from N^s to N_1^s , because leisure is a normal good



The economist's problem

- ▶ You have a model with n endogenous variables \mathbf{y} and m exogenous variables \mathbf{x} , whose solution is described by $\mathbf{y} = \Psi(\mathbf{x})$.
- ▶ You have found n model conditions of the form $g(\mathbf{x}, \mathbf{y}) = 0$.
- ▶ **Problem:** How to analyze the comparative statics of the model **without** an explicit formula for $\Psi(\mathbf{x})$?
- ▶ **Solution:** compute the total derivative of g , using the chain rule.

Side note:

Gradient, Jacobian, Hessian, and
total derivative

The gradient and the Hessian matrix

Let f be a function, $f : \mathbb{R}^n \rightarrow \mathbb{R}$, where $\mathbf{x} = (x_1 \ \cdots \ x_n)'$. We denote the first partial derivatives of $f(\mathbf{x})$ by

$$f_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i} \quad \text{and} \quad \nabla f(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

and the Hessian matrix of $f(\mathbf{x})$ by

$$H(\mathbf{x}) = \begin{bmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \cdots & f_{nn}(\mathbf{x}) \end{bmatrix}$$

The Jacobian

Let f be a function, $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$:

$$f(\mathbf{x}) = \begin{bmatrix} f^1(\mathbf{x}) \\ \vdots \\ f^m(\mathbf{x}) \end{bmatrix}$$

We denote **the Jacobian** of $f(\mathbf{x})$ by

$$J(\mathbf{x}) = \begin{bmatrix} f_1^1(\mathbf{x}) & f_2^1(\mathbf{x}) & \dots & f_n^1(\mathbf{x}) \\ f_1^2(\mathbf{x}) & f_2^2(\mathbf{x}) & \dots & f_n^2(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^m(\mathbf{x}) & f_2^m(\mathbf{x}) & \dots & f_n^m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \nabla f^1(\mathbf{x})' \\ \nabla f^2(\mathbf{x})' \\ \vdots \\ \nabla f^m(\mathbf{x})' \end{bmatrix}$$

A partitioned Jacobian

- ▶ Let $g(\mathbf{x}, \mathbf{y})$ be a function of vectors $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, such that $g : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$.
- ▶ Think of g as a system of n nonlinear equations on n endogenous variables \mathbf{y} and m exogenous variables \mathbf{x} .
- ▶ The partial Jacobians Dg_y and Dg_x form a partition of the Jacobian:

$$J(\mathbf{x}, \mathbf{y}) = [Dg_y \mid Dg_x] = \begin{bmatrix} g_{y_1}^1 & g_{y_2}^1 & \cdots & g_{y_n}^1 & g_{x_1}^1 & g_{x_2}^1 & \cdots & g_{x_m}^1 \\ g_{y_1}^2 & g_{y_2}^2 & \cdots & g_{y_n}^2 & g_{x_1}^2 & g_{x_2}^2 & \cdots & g_{x_m}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{y_1}^n & g_{y_2}^n & \cdots & g_{y_n}^n & g_{x_1}^n & g_{x_2}^n & \cdots & g_{x_m}^n \end{bmatrix}$$

The total derivative

- ▶ The total derivative of $g(\mathbf{x}, \mathbf{y})$ satisfies

$$\sum_{i=1}^n \frac{\partial g^k}{\partial y_i} dy_i + \sum_{i=1}^m \frac{\partial g^k}{\partial x_i} dx_i = 0, \quad \forall k = 1, \dots, n$$

- ▶ This can be written in terms of the partitioned Jacobian:

$$\begin{aligned} 0 &= \begin{bmatrix} g_{y_1}^1 & g_{y_2}^1 & \cdots & g_{y_n}^1 \\ g_{y_1}^2 & g_{y_2}^2 & \cdots & g_{y_n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ g_{y_1}^n & g_{y_2}^n & \cdots & g_{y_n}^n \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix} + \begin{bmatrix} g_{x_1}^1 & g_{x_2}^1 & \cdots & g_{x_m}^1 \\ g_{x_1}^2 & g_{x_2}^2 & \cdots & g_{x_m}^2 \\ \vdots & \vdots & \ddots & \vdots \\ g_{x_1}^n & g_{x_2}^n & \cdots & g_{x_m}^n \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix} \\ &= Dg_y \, d\mathbf{y} + Dg_x \, d\mathbf{x} \end{aligned}$$

- ▶ Then $d\mathbf{y} = -[Dg_y]^{-1} Dg_x \, d\mathbf{x}$, assuming inverse is defined.

Comparative statics in leisure-consumption model

In our leisure-consumption model, the solution required that:

$$g_1(c, l, w, \pi) = U_l - wU_c = 0$$

$$g_2(c, l, w, \pi) = c - wh + wl - \pi = 0$$

Therefore

$$\begin{aligned} 0 &= \begin{bmatrix} g_c^1 & g_l^1 \\ g_c^2 & g_l^2 \end{bmatrix} \begin{bmatrix} dc \\ dl \end{bmatrix} + \begin{bmatrix} g_w^1 & g_\pi^1 \\ g_w^2 & g_\pi^2 \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \\ &= \begin{bmatrix} U_{lc} - wU_{cc} & U_{ll} - wU_{cl} \\ 1 & w \end{bmatrix} \begin{bmatrix} dc \\ dl \end{bmatrix} + \begin{bmatrix} -U_c & 0 \\ l - h & -1 \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} dc \\ dl \end{bmatrix} &= \begin{bmatrix} U_{lc} - wU_{cc} & U_{ll} - wU_{cl} \\ 1 & w \end{bmatrix}^{-1} \begin{bmatrix} U_c & 0 \\ h-l & 1 \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \\
&= \frac{1}{\nabla} \begin{bmatrix} w & wU_{cl} - U_{ll} \\ -1 & U_{lc} - wU_{cc} \end{bmatrix} \begin{bmatrix} U_c & 0 \\ h-l & 1 \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \\
&= \frac{1}{\nabla} \begin{bmatrix} wU_c + (h-l)(wU_{cl} - U_{ll}) & wU_{cl} - U_{ll} \\ -U_c + (h-l)(U_{lc} - wU_{cc}) & U_{lc} - wU_{cc} \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix}
\end{aligned}$$

where

$$\nabla = - (w^2 U_{cc} - 2wU_{cl} + U_{ll}) = - \begin{vmatrix} 0 & 1 & w \\ 1 & U_{cc} & U_{cl} \\ w & U_{lc} & U_{ll} \end{vmatrix} \geq 0$$

The comparative statics follows from:

$$\frac{dc}{d\pi} = \frac{wU_{cl} - U_{ll}}{\nabla} > 0 \quad (c \text{ is normal})$$

$$\frac{dc}{dw} = \frac{wU_c + (h - l)(wU_{cl} - U_{ll})}{\nabla} > 0$$

$$\frac{dl}{d\pi} = \frac{U_{lc} - wU_{cc}}{\nabla} > 0 \quad (l \text{ is normal})$$

$$\frac{dl}{dw} = \frac{-U_c + (h - l)(U_{lc} - wU_{cc})}{\nabla} \quad ? \quad 0$$

Choice under uncertainty

Choice under uncertainty

- ▶ Until now, we have been concerned with the behavior of a consumer under conditions of certainty.
- ▶ However, many choices made by consumers take place under conditions of uncertainty.
- ▶ In this section we explore how the theory of consumer choice can be used to describe such behavior.

- ▶ The first question to ask is **what is the basic “thing” that is being chosen?**
- ▶ The consumer is presumably concerned with the probability distribution of getting different consumption bundles of goods.
- ▶ A probability distribution consists of a list of different outcomes—in this case, consumption bundles—and the probability associated with each outcome.
- ▶ When a consumer decides how much automobile insurance to buy or how much to invest in the stock market, he is in effect deciding on a pattern of probability distribution across different amounts of consumption.

Contingent consumption

- ▶ Let us think of the different outcomes of some random event as being different **states of nature**.
- ▶ A **contingent consumption plan** is a specification of what will be consumed in each different state of nature.
- ▶ Contingent means depending on something not yet certain.
- ▶ People have preferences over different plans of consumption, just like they have preferences over actual consumption.
- ▶ We can think of preferences as being defined over different consumption plans.

- ▶ If the consumer has reasonable preferences about consumption in different circumstances, then we can use a utility function to describe these preferences.
- ▶ However, **uncertainty** does add a special structure to the choice problem.
- ▶ How a person values consumption in one state as compared to another will depend on the probability that the state in question will actually occur.
- ▶ For this reason, we will write the utility function as depending on the probabilities as well as on the consumption levels.

Utility with discrete random outcomes

- ▶ If there are n possible states of nature s , then c is a **discrete random variable** with support $\{c_1, \dots, c_n\}$, whose values are realized with probabilities $\{p_1, \dots, p_n\}$.

s	\mathbb{P}	c	$u(c)$
1	π_1	c_1	$u(c_1)$
2	π_2	c_2	$u(c_2)$
\vdots		\vdots	
n	π_n	c_n	$u(c_n)$

- ▶ Utility is

$$U(c_1, \dots, c_n; \pi_1, \dots, \pi_n) = \sum_{i=1}^n \pi_i u(c_i)$$

Utility with continuous random outcomes

- ▶ If there are infinite states of nature, we think of c as a **continuous random variable**.
- ▶ If c has support \mathbf{C} , pdf $f(c)$ and cdf $F(c)$, then utility is

$$\begin{aligned}U(c, f) &= \int_{\mathbf{c}} f(c)u(c) \, dc \\ &= \int_{\mathbf{c}} u(c) \, dF(c)\end{aligned}$$

- ▶ We refer to a utility function U with the particular form described here as an **expected utility** function, or, sometimes, a **von Neumann-Morgenstern utility** function:

$$U(c, \mathbb{P}) \equiv \mathbb{E} u(c) = \begin{cases} \sum_{i=1}^n \pi_i u(c_i) & \text{discrete} \\ \int_{\mathbf{c}} u(c) \, dF(c) & \text{continuous} \end{cases}$$

- ▶ We refer to $u(c)$ as the **Bernoulli utility** function.

Growing potatoes in uncertain weather

- ▶ A farmer grows potatoes for own consumption.
- ▶ The weather s can be *good* or *bad*, affecting the amount of potatoes (real income y) he actually harvests:

s (weather)	\mathbb{P}	y
g (good)	π_g	W
b (bad)	π_b	$W - L$

- ▶ That is, if weather is bad, he loses L potatoes.
- ▶ Expected consumption of potatoes:

$$\mathbb{E} c = \mathbb{E} y = (1 - \pi_b)W + \pi_b(W - L) = W - \pi_b L$$

An insurance contract

- ▶ Farmer can insure K potatoes, premium is γ per unit.
- ▶ Choices are contingent consumption plans:

s	\mathbb{P}	y	insure	c
g	π_g	W	$-\gamma K$	$W - \gamma K$
b	π_b	$W - L$	$(1 - \gamma)K$	$W - \gamma K + K - L$

Expected utility of buying insurance coverage K

- ▶ Expected utility is

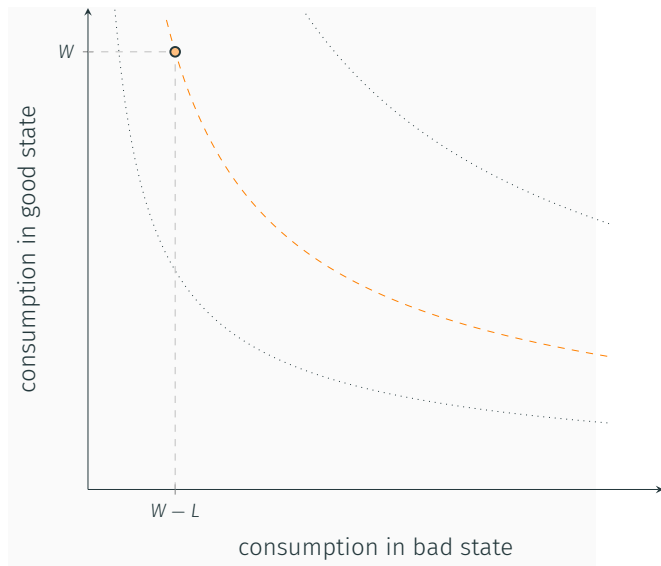
$$\begin{aligned}U(c_g, c_b; \pi_g, \pi_b) &\equiv \mathbb{E} u(c) \\&= \pi_g u(c_g) + \pi_b u(c_b) \\&= \pi_g u(W - \gamma K) + \pi_b u(W - \gamma K + K - L)\end{aligned}$$

- ▶ MRS of bad-weather potatoes for one good-weather potato is

$$MRS_{bg} = \frac{U_{c_b}}{U_{c_g}} = \frac{\pi_b u'(c_b)}{\pi_g u'(c_g)}$$

Objective function:

$$\mathbb{E} u(c) = U(c_g, c_b, \pi_g, \pi_b) = \pi_g u(c_g) + \pi_b u(c_b)$$



Budget constraint $(c_g, c_b) = (y_g - \gamma K, y_b + (1 - \gamma)K)$

- ▶ We have

$$K = \frac{y_g - c_g}{\gamma} = \frac{c_b - y_b}{1 - \gamma}$$

- ▶ Therefore

$$c_g + \frac{\gamma}{1 - \gamma} c_b = y_g + \frac{\gamma}{1 - \gamma} y_b$$

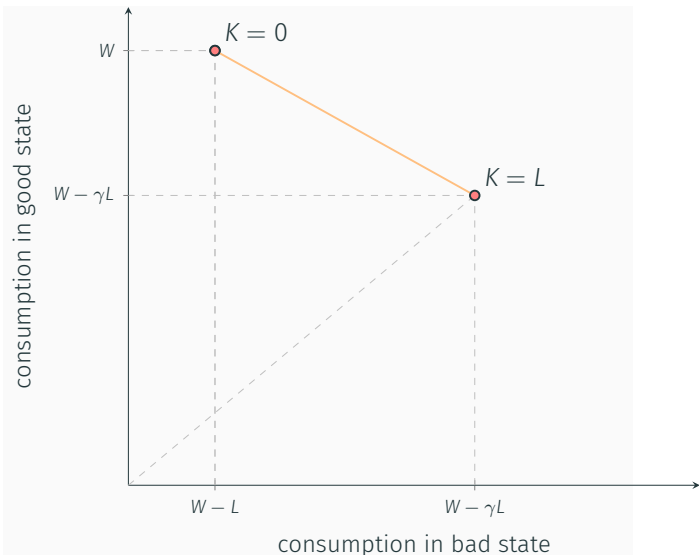
- ▶ Substitute $y_g = W$ and $y_b = W - L$ to get

$$\begin{aligned} c_g + \frac{\gamma}{1 - \gamma} c_b &= W + \frac{\gamma}{1 - \gamma} (W - L) \\ &= \frac{1}{1 - \gamma} W - \frac{\gamma}{1 - \gamma} L \end{aligned}$$

- ▶ The relative price (in terms of potatoes in good weather) of a potato in bad weather is $p = \frac{\gamma}{1 - \gamma}$

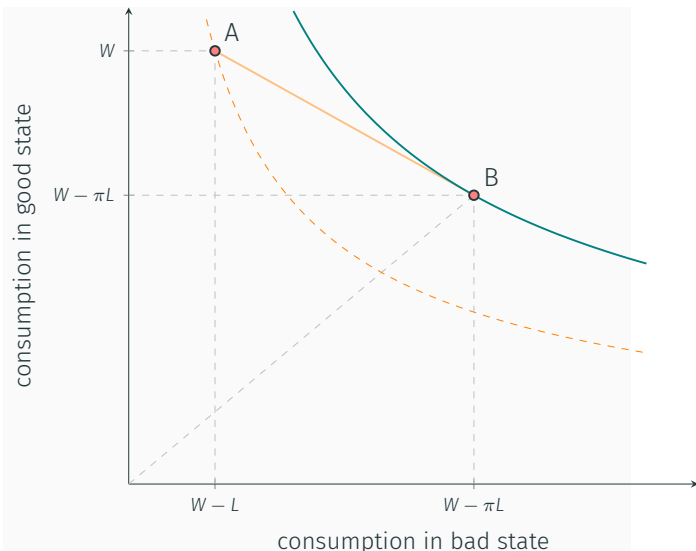
Budget constraint:

$$c_g + \frac{\gamma}{1-\gamma}c_b = \frac{1}{1-\gamma}W - \frac{\gamma}{1-\gamma}L$$



Optimality condition:

$$MRS_{bg} = \frac{\pi_b u'(c_b)}{\pi_g u'(c_g)} = \frac{\gamma}{1 - \gamma} = p$$



- ▶ Of course, we could also solve for optimal K directly:

$$\max_K \{ \pi_g u(W - \gamma K) + \pi_b u(W - L - \gamma K + K) \}$$

- ▶ FOC:

$$0 = -\gamma \pi_g u'(W - \gamma K) + (1 - \gamma) \pi_b u'(W - L - \gamma K + K)$$

$$\Leftrightarrow \frac{\pi_b u'(W - L - \gamma K + K)}{\pi_g u'(W - \gamma K)} = \frac{\gamma}{1 - \gamma}$$

Risk of losses and price of insurance

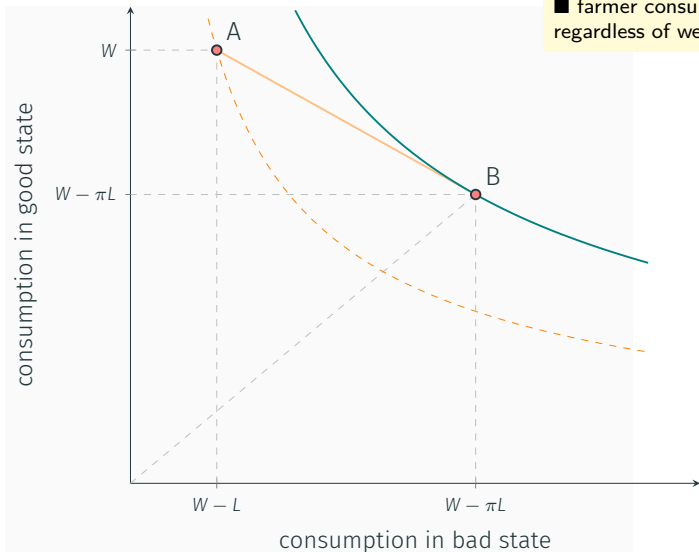
- ▶ The market price of insurance should satisfy $\gamma \geq \pi_b$, so the insurer gets enough revenue γK to cover expected payments $\pi_b K$. This implies that:

$$\begin{aligned}\gamma &\geq \pi_b \\ 1 - \pi_b &\geq 1 - \gamma \\ \gamma(1 - \pi_b) &\geq \pi_b(1 - \gamma) \\ 1 &\geq \frac{\pi_b(1 - \gamma)}{\gamma(1 - \pi_b)} = \frac{u'(c_g)}{u'(c_b)} && \text{(from FOC)} \\ u'(c_b) &\geq u'(c_g) \\ c_b &\leq c_g && \text{(assuming risk aversion)}\end{aligned}$$

- ▶ Consumer gets full insurance iff it's actuarially fair.

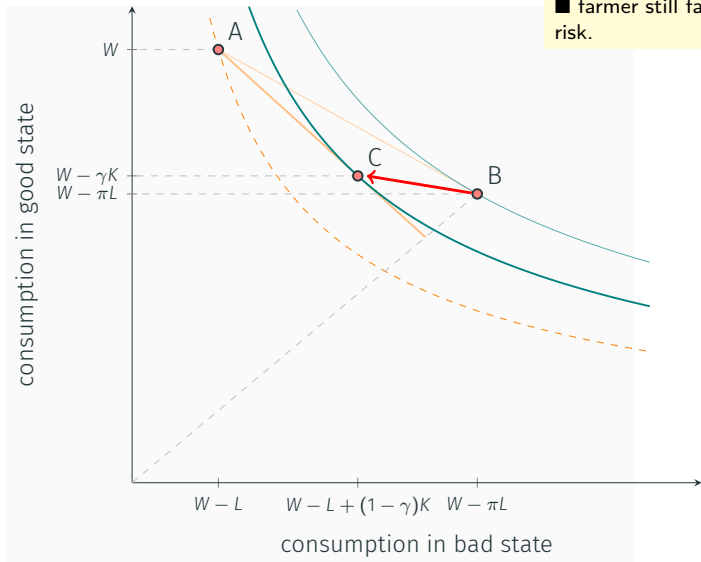
Case $\gamma = \pi_b$: actuarially fair insurance

- $K^* = L$: full insurance
- farmer consumes $W - \pi_b L$, regardless of weather.

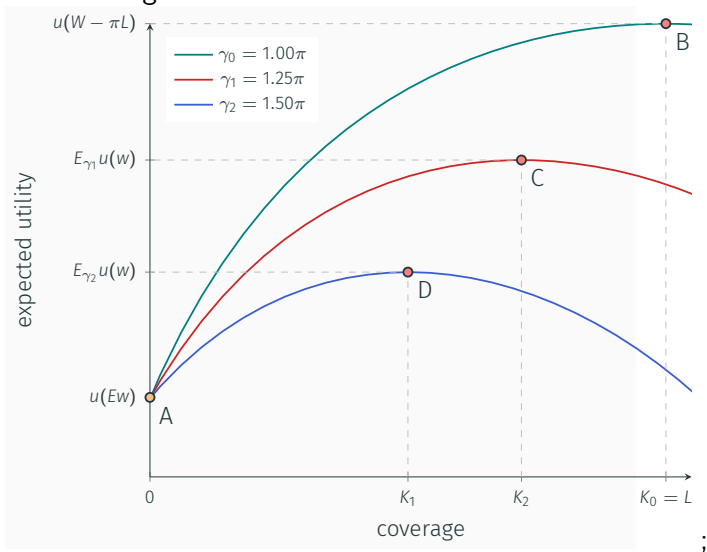


Case $\gamma > \pi_b$: insurer expects a profit

- $K^* < L$: no full insurance
- farmer still faces (lower) risk.



Increasing the premium: As insurance gets expensive, consumer buys less coverage.



Example 1:

Logarithmic utility

Let's now assume that $u(c) = \ln(c)$ From the FOC:

$$\gamma\pi_g u'(W - \gamma K) = (1 - \gamma)\pi_b u'(W - L + (1 - \gamma)K)$$

$$\gamma\pi_g [W - L + (1 - \gamma)K] = (1 - \gamma)\pi_b (W - \gamma K)$$

$$\pi_g \gamma (W - L) + \pi_g \gamma (1 - \gamma) K = \pi_b (1 - \gamma) W - \pi_b \gamma (1 - \gamma) K$$

$$(\pi_b + \pi_g)(1 - \gamma)\gamma K = (\pi_b - \gamma(\pi_b + \pi_g))W + \gamma\pi_g L$$

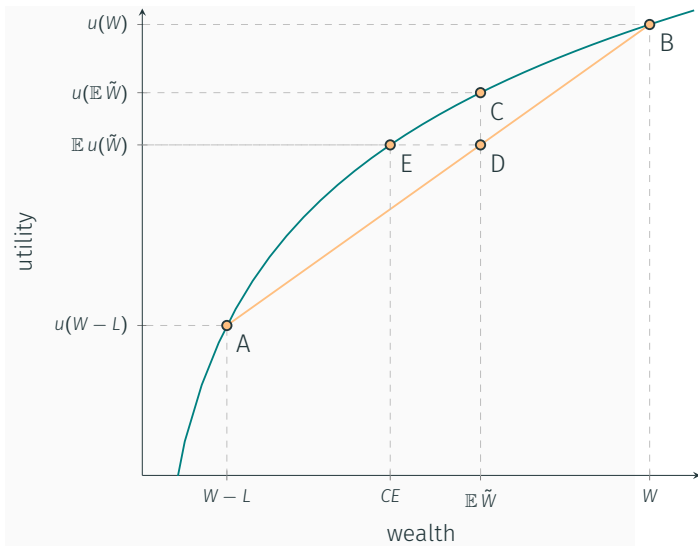
$$\gamma(1 - \gamma)K = (\pi_b - \gamma)W + \gamma(1 - \pi_b)L$$

$$K^* = \frac{1 - \pi_b}{1 - \gamma} L - \frac{\gamma - \pi_b}{\gamma(1 - \gamma)} W$$

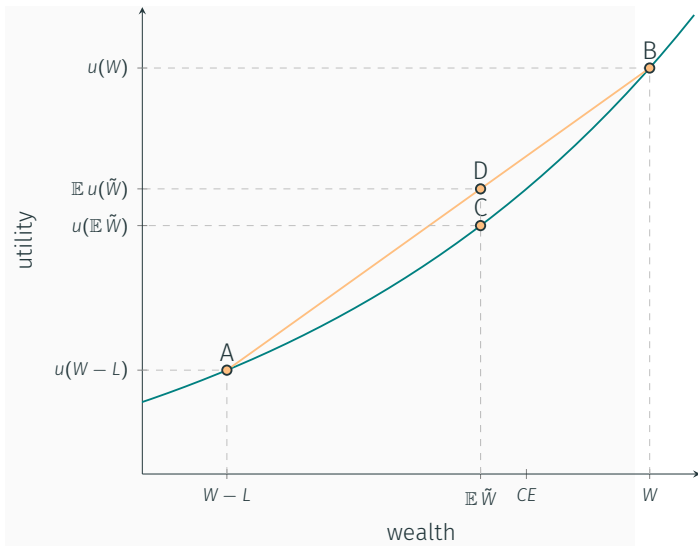
Optimal contingent consumption plans:

s	\mathbb{P}	y	c^*
g	π_g	W	$\frac{1-\pi_b}{1-\gamma}(W - \gamma L)$
b	π_b	$W - L$	$\frac{\pi_b}{\gamma}(W - \gamma L)$

A risk averse consumer



A risk loving consumer



Measuring risk aversion

- ▶ A consumer with a von Neumann-Morgenstern utility function can be one of the following:
 - ▶ Risk-averse, with a concave utility function;
 - ▶ Risk-neutral, with a linear utility function, or;
 - ▶ Risk-loving, with a convex utility function.
- ▶ Then, the degree of risk-aversion a consumer displays would be related to the curvature of their Bernoulli utility function $u(W)$.
- ▶ The more “curved” a concave $u(W)$ is, the lower will be a consumer’s certainty equivalent, and the higher their risk premium.
- ▶ How do we measure the curvature of a function?
- ▶ Simple - using the function’s second derivative.

Absolute risk aversion

$$\frac{-u''(W)}{u'(W)}$$

Relative risk aversion

$$\frac{-u''(W)W}{u'(W)}$$

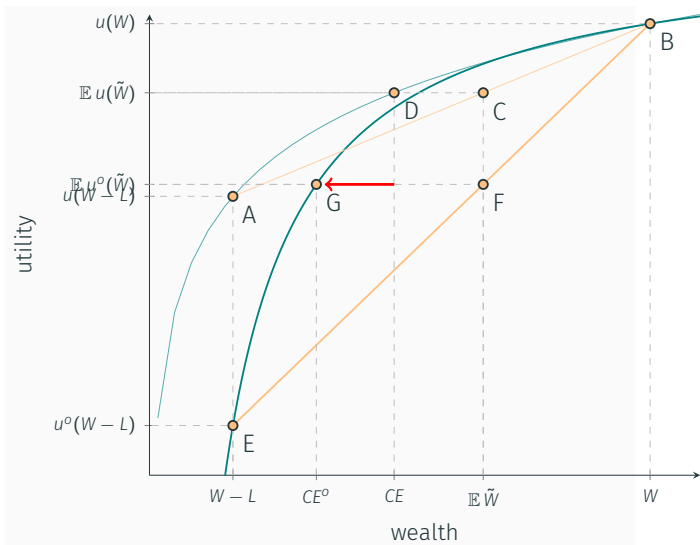
CARA

$$u(c) = -e^{-\rho c}$$

CRRA

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

A change in risk aversion



A risky asset

- ▶ Consider a simple two-period portfolio problem involving two assets, one with a risky (gross) return $\tilde{R} \geq 0$ and one with a sure (gross) return $R_f \geq 1$.
- ▶ Let w be initial wealth, and let $x \in [0, 1]$ be the share of wealth invested in the risky asset.

s	\mathbb{P}	risky	risk-free	c
$\tilde{R} = R$	$f(R)$	xw	$(1-x)w$	$[(1-x)R_f + xR]w$

- ▶ In this case the second-period wealth can be written as

$$\begin{aligned}\tilde{W} &= (1-x)R_f w + x\tilde{R}w \\ &= [(1-x)R_f + x\tilde{R}]w\end{aligned}$$

- ▶ Note that \tilde{W} is a random variable since \tilde{R} is random.

- ▶ The expected utility from investing x in the risky asset:

$$v(x) = \mathbb{E} u(c) = \mathbb{E} u \left([(1-x)R_f + x\tilde{R}]w \right)$$

- ▶ The portfolio problem is then to choose $x \in [0, 1]$ to maximize $v(x)$:

$$\mathcal{L}(x, \mu, \lambda) = \mathbb{E} u \left([(1-x)R_f + x\tilde{R}]w \right) + \mu x + \lambda(1-x)$$

- ▶ Conditions:

$$\mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f)w \right\} + \mu - \lambda = 0$$

$$\mu \geq 0 \qquad x \geq 0 \qquad \mu x = 0$$

$$\lambda \geq 0 \qquad x \leq 1 \qquad \lambda(1-x) = 0$$

Second order condition

- ▶ Notice that second derivative is

$$\mathbb{E} \left\{ u''(\tilde{W})(\tilde{R} - R_f)^2 w^2 \right\} < 0 \quad \text{iif} \quad u''(\tilde{W}) < 0$$

- ▶ SOC requires that consumer is risk-averse.

Slackness conditions

- ▶ The slackness conditions (SC) imply:
 - ▶ if $x = 0$, 2nd group of SC satisfied with $\lambda = 0$.
 - ▶ if $x = 1$, 1st group of SC satisfied with $\mu = 0$.
 - ▶ if $0 < x < 1$, both groups of SC satisfied with $\lambda = \mu = 0$.

- ▶ Then, we only need to analyze 3 cases:
 - ▶ $x = 0 \Rightarrow \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} = -\mu \leq 0$
 - ▶ $x = 1 \Rightarrow \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} = \lambda \geq 0$
 - ▶ $0 < x < 1 \Rightarrow \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} = 0$

Case 1: $x = 0 \Rightarrow \tilde{W} = wR_f$

$$\mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} \leq 0$$

$$\mathbb{E} \left\{ u'(\tilde{W})\tilde{R} \right\} \leq \mathbb{E} \left\{ u'(\tilde{W})R_f \right\}$$

$$\mathbb{E} \left\{ u'(wR_f)\tilde{R} \right\} \leq \mathbb{E} \left\{ u'(wR_f)R_f \right\}$$

$$u'(wR_f) \mathbb{E} \left\{ \tilde{R} \right\} \leq u'(wR_f)R_f$$

$$\mathbb{E} \left\{ \tilde{R} \right\} \leq R_f$$

Consumer does not invest in risky asset if its expected return is lower than the risk-free return.

Case 2: $x = 1 \Rightarrow \tilde{W} = w\tilde{R}$

$$\begin{aligned}\mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} &\geq 0 \\ \mathbb{E} \left\{ u'(\tilde{W})\tilde{R} \right\} &\geq \mathbb{E} \left\{ u'(\tilde{W})R_f \right\} \\ \mathbb{E} \left\{ u'(w\tilde{R})\tilde{R} \right\} &\geq \mathbb{E} \left\{ u'(w\tilde{R})R_f \right\} \\ R_f &\leq \frac{\mathbb{E} \left\{ u'(w\tilde{R})\tilde{R} \right\}}{\mathbb{E} \left\{ u'(w\tilde{R}) \right\}}\end{aligned}$$

Consumer does not invest in risk-free asset if its return is “too low”. We need more details about the \tilde{R} process and utility u to determine what “too low” is.

Case 3: $0 < x < 1$

$$\begin{aligned} 0 &= \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} \\ &= \text{Cov} \left[u'(\tilde{W}), \tilde{R} - R_f \right] + \mathbb{E} \left[u'(\tilde{W}) \right] \mathbb{E} \left[\tilde{R} - R_f \right] \end{aligned}$$

Then

$$\mathbb{E} \tilde{R} - R_f = \frac{-\text{Cov} \left[u'(\tilde{W}), \tilde{R} \right]}{\mathbb{E} u'(\tilde{W})} > 0$$

Example 2:

“Investing” in “Tiempos”

- ▶ In “Tiempos” lottery, you pick one number out of 100, all of them with equal probability (1%) of winning.
- ▶ In *winning* state, your gross return is $\tilde{R} = 72$.
- ▶ In *losing* state, your gross return is $\tilde{R} = 0$.
- ▶ If you don't play, you keep your money ($R_f = 1$).
- ▶ Expected return on lottery is

$$\mathbb{E} \tilde{R} = 0.99 \times 0 + 0.01 \times 72 = 0.7128 < 1 = R_f$$

- ▶ Therefore, a risk-averse consumer would never play “Tiempos”.

Intertemporal consumption

Adding a time dimension

- ▶ So far we have only studied static choices.
- ▶ Life is full of intertemporal choices: Should I study for my test today or tomorrow? Should I save or should I consume now?
- ▶ We will present a simple model: the Life-Cycle/Permanent Income Model of Consumption.
- ▶ Developed by Modigliani (Nobel winner 1985) and Friedman (Nobel winner 1976).
- ▶ Will allow us to address several key issues: effects of government programs including Social Security, government debts and deficits.

- ▶ Representative household lives 2 periods.
- ▶ Utility function:

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

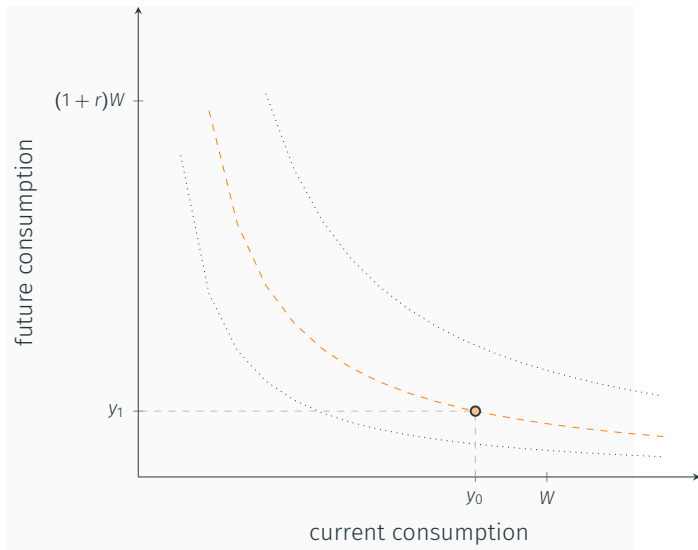
- ▶ c_0 is consumption in first (current) period of life,
 - ▶ c_1 is consumption in second (future) period of life,
 - ▶ $0 < \beta < 1$ measures household's degree of impatience.
- ▶ Preferences over c_0, c_1 satisfy monotonicity ($u' > 0$) and convexity ($u'' < 0$).

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

- ▶ Consumption smoothing motive, partially offset by discounting.
- ▶ Assume c_0 and c_1 are normal: more income \Rightarrow more of both.
- ▶ Intertemporal marginal rate of substitution measures willingness to substitute consumption over time:

$$MRS_{c_0, c_1} = \frac{U_{c_0}(c_0, c_1)}{U_{c_1}(c_0, c_1)} = \frac{u'(c_0)}{\beta u'(c_1)}$$

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$



Budget constraint I

- ▶ Abstract from labor/leisure tradeoff.
- ▶ (Labor) income $y_t \geq 0$ in period $t = 0, 1$.
- ▶ Initial wealth $a_0 \geq 0$.
- ▶ Consumer can save part of income or initial wealth in the first period, or it can borrow against future income y_1 .
- ▶ Interest rate on both savings and on loans is equal to r . Gross interest rate $R \equiv 1 + r$
- ▶ Let $s_t = y_t - c_t$ denote saving.
- ▶ Budget constraint in first period:

$$a_1 = R(a_0 + s_0)$$

- ▶ Budget constraint in second period:

$$a_2 = R(a_1 + s_1) = 0$$

Budget constraint (II)

- ▶ Combining both constraints:

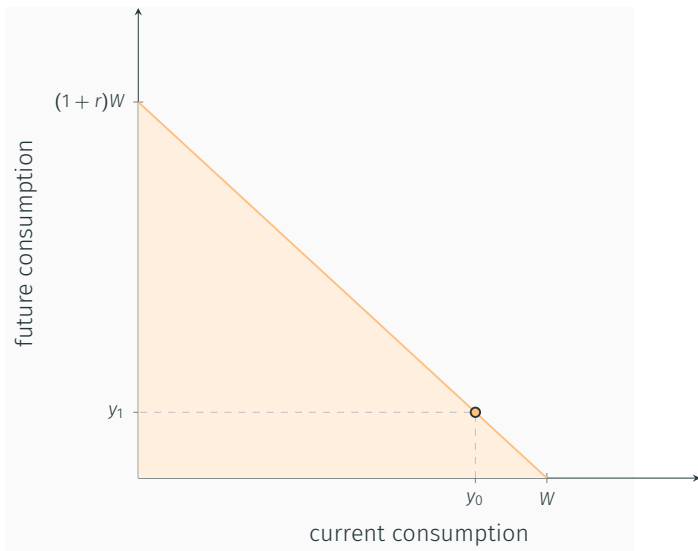
$$R(a_0 + s_0) + s_1 = 0 \quad \Rightarrow \quad -s_0 - \frac{s_1}{R} = a_0$$

- ▶ Substitute $s_t = y_t - c_t$

$$c_0 + \frac{c_1}{R} = y_0 + \frac{y_1}{R} + a_0 = H + a_0 \equiv W \quad (\text{PVBC})$$

- ▶ We have normalized the price of the consumption good in the first period to 1.
- ▶ Gross interest rate $R \equiv 1 + r$ is the relative price of consumption goods today to consumption goods tomorrow.
- ▶ Called the **present value budget constraint** (PVBC).

$$c_0 + \frac{c_1}{R} = W$$



The consumer's problem

$$\max_{c_0, c_1} \{u(c_0) + \beta u(c_1)\} \quad \text{subject to } c_0 + \frac{c_1}{R} = W$$

- ▶ Form Lagrangian with multiplier $\lambda \geq 0$

$$\mathcal{L}(c_0, c_1, \lambda) = u(c_0) + \beta u(c_1) + \lambda \left(W - c_0 - \frac{c_1}{R} \right)$$

- ▶ FOCs:

$$u'(c_0) = \lambda$$

$$\beta u'(c_1) = \frac{\lambda}{R}$$

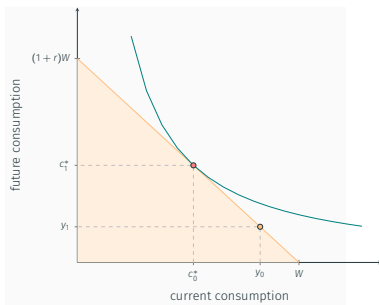
- ▶ Combine to get

Euler equation

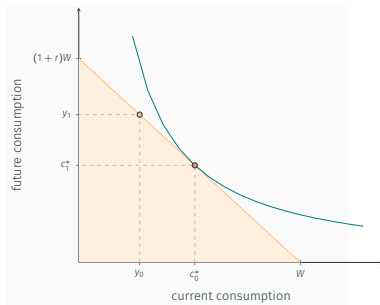
$$u'(c_0) = \beta R u'(c_1)$$

$$u'(c_0) = \beta R u'(c_1)$$

Consumer is a lender



Consumer is a borrower



Implications of the Euler equation

$$u'(c_0) = \beta R u'(c_1)$$

- ▶ Can also be written

$$MRS_{c_0, c_1} = 1 + r$$

- ▶ Recall that u is concave, so $u'' < 0 \Rightarrow u'(c)$ is decreasing. So if:
 - ▶ $\beta(1+r) > 1 \Rightarrow u'(c_0) > u'(c_1) \Rightarrow c_0 < c_1$
 - ▶ $\beta(1+r) < 1 \Rightarrow u'(c_0) < u'(c_1) \Rightarrow c_0 > c_1$
 - ▶ $\beta(1+r) = 1 \Rightarrow u'(c_0) = u'(c_1) \Rightarrow c_0 = c_1$
- ▶ Behavior of consumption over time depends on rate of time preference relative to interest rate.
- ▶ If equal, perfect consumption smoothing.

Example 3:

Logarithmic utility

$$u(c) = \ln(c)$$

- ▶ Euler equation:

$$\frac{1}{c_0} = \frac{\beta R}{c_1} \quad \Rightarrow \quad c_1 = \beta R c_0$$

- ▶ Using the PVBC

$$c_0 = W - \frac{c_1}{R} = W - \beta c_0$$

- ▶ So that

$$\begin{aligned} c_0 &= \frac{1}{1+\beta} W & s_0 &= \frac{1}{1+\beta} \left(\beta y_0 - a_0 - \frac{y_1}{R} \right) \\ c_1 &= \frac{\beta R}{1+\beta} W & a_1 &= \frac{1}{1+\beta} [\beta R(y_0 + a_0) - y_1] \end{aligned}$$

- ▶ Value function:

$$V(W, r) = (1 + \beta) \ln W + \beta \ln R + \beta \ln \beta - (1 + \beta) \ln(1 + \beta)$$

- ▶ Increasing wealth W , regardless of source, increases consumer utility:

$$\frac{\partial V}{\partial W} = \frac{1 + \beta}{W}$$

- ▶ Effect of a change in interest rate r depends on wealth composition, which in turn determines whether the consumer has positive or negative assets a_1 at the end of period 1:

$$\begin{aligned} \frac{\partial V}{\partial r} &= \frac{1}{R^2 W} [\beta R(y_0 + a_0) - y_1] \\ &= \frac{1 + \beta}{R^2 W} a_1 \end{aligned}$$

Example 4:
CRRA utility

- ▶ The logarithmic utility from last example is just a special case of the constant relative risk aversion (CRRA) utility, when $\sigma = 1$.

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

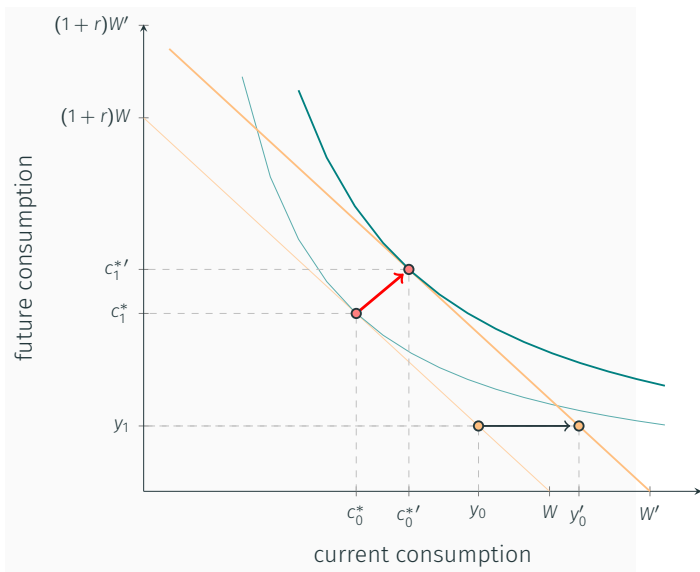
- ▶ With CRRA utility, the Bellman equation becomes

$$c_0^{-\sigma} = \beta R c_1^{-\sigma} \quad \Rightarrow \quad c_1 = (\beta R)^{1/\sigma} c_0$$

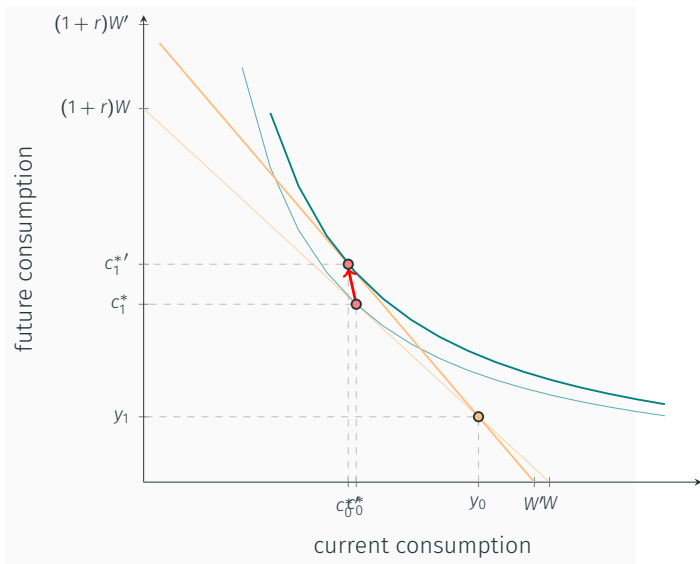
- ▶ Use budget constraint $c_0 + \frac{c_1}{R} = W$ to solve for c_0 and c_1 :

$$c_0 = \frac{R}{R + (\beta R)^{1/\sigma}} W \quad c_1 = \frac{R(\beta R)^{1/\sigma}}{R + (\beta R)^{1/\sigma}} W$$

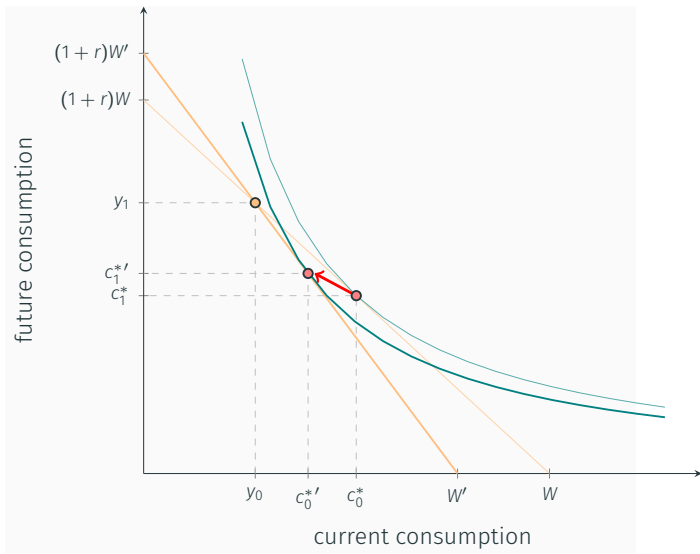
Increasing wealth



Increasing interest rate: lender



Increasing interest rate: borrower



The model

- ▶ A consumer lives two periods, and chooses among $n + 1$ goods in each period: x_{it} for $i \in \{0, 1, \dots, n\}$ and $t \in \{0, 1\}$.
- ▶ Utility function depends on $2n + 2$ goods:

$$U = \frac{(\alpha_0 x_{00}^\rho + \dots + \alpha_n x_{n0}^\rho)^{\frac{1-\gamma}{\rho}}}{1-\gamma} + \beta \frac{(\alpha_0 x_{01}^\rho + \dots + \alpha_n x_{n1}^\rho)^{\frac{1-\gamma}{\rho}}}{1-\gamma}$$

- ▶ Let \mathbf{x}_t be the bundle of goods consumed at time t :

$$\mathbf{x}_t = [x_{0t}, x_{1t}, \dots, x_{nt}]$$

Constraints in nominal terms

- ▶ Consumer can save and borrow money at nominal interest rate i .
- ▶ The budget constraint says that the present value of all consumption purchases must equal the present value of nominal income Y_t :

$$\sum_{k=0}^n p_{k0}x_{k0} + \frac{1}{1+i} \sum_{k=0}^n p_{k1}x_{k1} = Y_0 + \frac{Y_1}{1+i}$$

- ▶ Let $C_t = \sum_{k=0}^n p_{kt}x_{kt}$ be nominal consumption at time t .
- ▶ Budget constraint becomes

$$C_0 + \frac{C_1}{1+i} = Y_0 + \frac{Y_1}{1+i} \equiv W$$

where W is nominal wealth.

Constraints in real terms

- ▶ Let $P_t = (\alpha_0^\sigma p_{0t}^{1-\sigma} + \dots + \alpha_n^\sigma p_{nt}^{1-\sigma})^{\frac{1}{1-\sigma}}$ be the price index at time t
- ▶ Notice that $\frac{P_1}{P_0(1+i)} = \frac{1+\pi}{1+i} = \frac{1}{1+r}$, where π is the inflation rate, and r the real interest rate.
- ▶ Divide budget constraint by price index P_0

$$\frac{C_0}{P_0} + \frac{P_1}{P_0(1+i)} \frac{C_1}{P_1} = \frac{Y_0}{P_0} + \frac{P_1}{P_0(1+i)} \frac{Y_1}{P_1} = \frac{W}{P_0}$$
$$c_0 + \frac{c_1}{1+r} = y_0 + \frac{y_1}{1+r} = w$$

where c_t is real consumption, y_t is real income, and w is real wealth.

- ▶ Constraint says that present value of real (composite) consumption equals the present value of real income.

Solving the problem: 2 steps

- ▶ Let \tilde{U} denote CES function: $\tilde{U}(\mathbf{x}_t) = (\alpha_0 x_{0t}^\rho + \dots + \alpha_n x_{nt}^\rho)^{\frac{1}{\rho}}$
- ▶ Utility becomes:

$$U = \frac{\tilde{U}(\mathbf{x}_0)^{1-\gamma}}{1-\gamma} + \beta \frac{\tilde{U}(\mathbf{x}_1)^{1-\gamma}}{1-\gamma}$$

- ▶ Consumer has to choose $2n + 2$ variables, subject to 1 budget constraint.
- ▶ To solve this problem, consumer makes decisions in two stages
 - ▶ **Intra-temporal stage:** Given all prices and the total amount to spend in each period, consumer chooses goods for each period separately.
 - ▶ **Inter-temporal stage:** Taking the intra-temporal solution as given, solve the inter-temporal problem:

- ▶ Given all prices and the total amount to spend in each period, consumer chooses goods for each period separately.
- ▶ Since intra-temporal preferences are CES, we know (from Example 4 in Lecture 7) that if consumer spends C_t dollars and price level is P_t , the optimal utility he can get is

$$\tilde{V}(C_t, P_t) \equiv \max_{\mathbf{x}_t} \tilde{U}(\mathbf{x}_t) = \frac{C_t}{P_t} = c_t$$

- ▶ Taking the intra-temporal solution as given, problem becomes:

$$\max_{c_0, c_1} \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \frac{c_1^{1-\gamma}}{1-\gamma} \quad \text{s.t.} \quad c_0 + \frac{c_1}{R} = w$$

- ▶ But this is equivalent to what we solved in Example 4 in this lecture. Its solution is characterized by the **Euler equation**

$$c_0^{-\gamma} = \beta R c_1^{-\gamma} \quad \Rightarrow \quad c_1 = (\beta R)^{1/\gamma} c_0$$

- ▶ Solution is

$$c_0 = \frac{R}{R + (\beta R)^{1/\gamma}} w \quad c_1 = \frac{R(\beta R)^{1/\gamma}}{R + (\beta R)^{1/\gamma}} w$$

- ▶ Demands for each of the goods in then:

$$\begin{aligned}x_{kt} &= \left(\frac{\alpha_k}{\frac{p_{kt}}{P_t}} \right) c_t \\ &= \left(\frac{\alpha_k}{\frac{p_{kt}}{P_t}} \right) \frac{R(\beta R)^{t/\gamma}}{R + (\beta R)^{1/\gamma}} w\end{aligned}$$

- ▶ Notice that demand for goods depends only on preference parameters (α_k) and real variables (wealth w , interest rates r , relative prices p_{kt}/P_t)

Modeling implications

- ▶ If utility is time-separable, we can split the problem of choosing n goods over T periods into $T + 1$ problems:
 - ▶ decide how much to spend in each of the T periods (inter-temporal allocation); and
 - ▶ take each period budget and decide how to spend it into the n goods (intra-temporal allocation)
- ▶ If intra-temporal preference is CES, we can interpret the indirect utilities of the intra-temporal allocations as **real composite consumption good**.
- ▶ From now on, in our macro models we will analyze dynamic consumption behavior assuming that there exist such real composite consumption good.
- ▶ We will simply call it the consumption good.

Intertemporal consumption with uncertainty

Intertemporal consumption with uncertainty

- ▶ Representative consumer lives 2 periods.
- ▶ She can save and borrow at interest rate r .
- ▶ Her initial asset is a_0 .
- ▶ She doesn't leave any debt or inheritance ($a_2 = 0$).
- ▶ Her income $y_t \geq 0$ in period $t = 0, 1$:
 - ▶ y_0 is known at time of deciding c_0 .
 - ▶ \tilde{y}_1 is uncertain. It takes value y_{1s} with probability π_s , depending on the state of nature $s = 1, \dots, S$.
 - ▶ Notice that $\sum_{s=1}^S \pi_s = 1$.
- ▶ Her **expected** future income is then

$$\mathbb{E} \tilde{y}_1 = \sum_{s=1}^S \pi_s y_{1s}$$

- ▶ Budget constraints:

$$a_1 = R(a_0 + y_0 - c_0)$$

$$a_2 = R(a_1 + \tilde{y}_1 - \tilde{c}_1) = 0$$

- ▶ a_0 and y_0 are certain (she already have them in her bank).
- ▶ c_0 and a_1 are certain (she nows what she is choosing **now**).
- ▶ c_1 is uncertain because she needs to adjust future consumption to income shocks:

$$\tilde{c}_1 = a_1 + \tilde{y}_1 \quad \Rightarrow$$

$$\mathbb{E} \tilde{c}_1 = a_1 + \mathbb{E} \tilde{y}_1 \quad \Rightarrow$$

$$\tilde{c}_1 = \mathbb{E} \tilde{c}_1 + \underbrace{\tilde{y}_1 - \mathbb{E} \tilde{y}_1}_{\text{forecast error}}$$

Consumption plans, contingent on income

State	\mathbb{P}	Period 0	Period 1
s	π_s	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_{1s} = a_1 + y_{1s}$

Example 5:

Only two states of nature

State	Probability	Period 0	Period 1
L	π_L	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_1^L = a_1 + y_1^L$
H	π_H	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_1^H = a_1 + y_1^H$

Consumer wants to maximize her discounted expected utility:

$$\begin{aligned}
 U(c_0, c_1^L, c_1^H, \pi_L, \pi_H) &= \mathbb{E}_{\tilde{y}_2} [u(c_0) + \beta u(c_1)] \\
 &= \pi_L [u(c_0) + \beta u(c_1^L)] + \pi_H [u(c_0) + \beta u(c_1^H)] \\
 &= (\pi_L + \pi_H)u(c_0) + \beta [\pi_L u(c_1^L) + \pi_H u(c_1^H)] \\
 &= u(c_0) + \beta \mathbb{E} u(c_1)
 \end{aligned}$$

$$\begin{aligned}
U &= \{u(c_0) + \beta \mathbb{E} u(c_1)\} \\
&= \{u(c_0) + \beta [\pi_L u(c_1^L) + \pi_H u(c_1^H)]\} \\
&= \left\{ u \left(a_0 + y_0 - \frac{a_1}{R} \right) + \beta [\pi_L u(a_1 + y_1^L) + \pi_H u(a_1 + y_1^H)] \right\}
\end{aligned}$$

Objective now depends on a_1 alone. Take FOC:

$$\begin{aligned}
0 &= -\frac{1}{R} u'(c_0) + \beta \pi_L u'(c_1^L) + \beta \pi_H u'(c_1^H) \\
u'(c_0) &= \beta R [\pi_L u'(c_1^L) + \pi_H u'(c_1^H)] \\
&= \beta R \mathbb{E} [u'(c_1)] \qquad \text{(Euler equation)}
\end{aligned}$$

Wealth and permanent income

- ▶ Combining the budget constraints she gets

$$c_0 + \frac{\tilde{c}_1}{R} = a_0 + \underbrace{y_0 + \frac{\tilde{y}_1}{R}}_{\text{wealth } \tilde{W}_0} \quad (\text{for any possible state of nature})$$

- ▶ Her wealth at time 0 is uncertain because future income is random. But she can form an expectation:

$$c_0 + \frac{\mathbb{E} \tilde{c}_1}{R} = a_0 + y_0 + \frac{\mathbb{E} \tilde{y}_1}{R} = \mathbb{E} \tilde{W}_0$$

- ▶ Her **permanent income** y_p is the constant level of consumption that she **expects** to be able to afford, given her **expected wealth**. Then

$$y_p = \frac{R}{1+R} \mathbb{E} \tilde{W}$$

The consumer's problem

- ▶ She wants to maximize her discounted expected utility (von Neumann-Morgenstern):

$$\begin{aligned}U\left(c_0, \{c_{1s}; \pi_s\}_{s=1}^S\right) &= \mathbb{E}_{\tilde{y}_2} [u(c_0) + \beta u(c_1)] \\ &= u(c_0) + \beta \mathbb{E} u(c_1)\end{aligned}$$

- ▶ subject to contingent plans

$$c_0 + \frac{c_{1s}}{R} = a_0 + y_0 + \frac{y_{1s}}{R} \equiv W_s \quad (\text{for } s = 1, \dots, S)$$

- ▶ There are S constraints (one per state of nature).
- ▶ Let $\lambda_s \pi_s$ be the Lagrange multiplier associated with the s^{th} constraint.

- ▶ The Lagrangian is

$$\begin{aligned}\mathcal{L} &= u(c_0) + \beta \mathbb{E} u(c_1) + \sum_s \lambda_s \pi_s \left(W_s - c_0 - \frac{c_{1s}}{R} \right) \\ &= u(c_0) + \sum_s \pi_s \left[\beta u(c_{1s}) + \lambda_s \left(W_s - c_0 - \frac{c_{1s}}{R} \right) \right]\end{aligned}$$

- ▶ FOCs:

$$\text{(wrt } c_0) \quad 0 = u'(c_0) - \sum_s \pi_s \lambda_s \quad \Rightarrow \quad u'(c_0) = \mathbb{E} \lambda$$

$$\text{(wrt } c_{1s}) \quad 0 = \pi_s \left[\beta u'(c_{1s}) - \frac{\lambda_s}{R} \right] \quad \Rightarrow \quad \pi_s \beta R u'(c_{1s}) = \pi_s \lambda_s$$

- ▶ Adding up the FOCs wrt c_{1s} , we get

$$\begin{aligned}\sum_s \pi_s \beta R u'(c_{1s}) &= \sum_s \pi_s \lambda_s \\ \beta R \mathbb{E} u'(c_1) &= \mathbb{E} \lambda\end{aligned}$$

- ▶ Substituting $\mathbb{E} \lambda$ from the first FOC to get

Euler equation

$$u'(c_0) = \beta R \mathbb{E} u'(c_1)$$

Side note:

Working with random variables

- ▶ Let u and v be functions, X and Z random variables, and a and b scalars.
- ▶ Suppose that X and Z depend on parameter t .
- ▶ Then, under fairly general conditions:

$$\mathbb{E}[au(X) + bv(Z)] = a \mathbb{E} u(X) + b \mathbb{E} v(Z)$$

$$\frac{\partial \mathbb{E} u(X)}{\partial t} = \mathbb{E} \left[u'(X) \frac{\partial X}{\partial t} \right]$$

A faster way to get the Euler equation

- ▶ Instead of having one constraint for each state of nature, just write one: the expected values of the constraint:

$$c_0 + \frac{\mathbb{E} \tilde{c}_1}{R} = \mathbb{E} \tilde{W}_0$$

- ▶ Just keep in mind that this is a shortcut: the budget constraint must be satisfied **in every state of nature**, not only in expected values.
- ▶ Besides, the consumer is choosing future consumption contingent on each state of nature. She is not just choosing her expected future consumption.

- ▶ Lagrangian is

$$\mathcal{L} = u'(c_0) + \beta \mathbb{E} u(c_1) + \lambda \left(\mathbb{E} \tilde{W} - c_0 - \frac{\mathbb{E} c_1}{R} \right)$$

- ▶ FOCs

$$\text{(wrt } c_0) \quad 0 = u'(c_0) - \lambda \quad \Rightarrow \quad u'(c_0) = \lambda$$

$$\text{(wrt } c_1) \quad 0 = \beta \mathbb{E} u'(c_1) - \frac{\lambda}{R} \quad \Rightarrow \quad \beta R \mathbb{E} u'(c_1) = \lambda$$

- ▶ Then, from the two FOCs

$$u'(c_0) = \beta R \mathbb{E} u'(c_1) \quad (\text{Euler equation})$$

- ▶ Euler equation can be written as:

$$\frac{u'(c_0)}{\beta \mathbb{E} u'(c_1)} = R$$

MRS of present
consumption for future
consumption

price of present
consumption in terms
of future consumption

Example 6:

Hall 1978

- ▶ Assume that utility is quadratic $u(c) = \alpha c - 0.5c^2$ and that $\beta R = 1$.
- ▶ Euler equation is:

$$\mathbb{E} c_1 = c_0$$

- ▶ This means that consumption would follow a **random walk**.
- ▶ In such case, under the pure life cycle-permanent income hypothesis, a forecast of future consumption obtained by extrapolating today's level by the historical trend is impossible to improve.

Example 7:

CRRA utility, with uncertainty

- ▶ Now assume that consumer has constant relative risk aversion: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 0$.
- ▶ Euler equation is:





$$c_0^{-\sigma} = \beta R \mathbb{E} (c_1^{-\sigma})$$

- ▶ But notice that $\mathbb{E} (c_1^{-\sigma}) \neq (\mathbb{E} c_1)^{-\sigma}$, so we can not simply use budget constraint

$$c_0 + \frac{\mathbb{E} \tilde{c}_1}{R} = \mathbb{E} \tilde{W}_0$$

to solve for c_0 and $\mathbb{E} c_1$.

- ▶ So, in dynamic models with uncertainty, it is often necessary to use numerical methods to analyze the solution of the model.

-  Hall, Robert E. (Dec. 1978). “Stochastic Implications of the Life Cycle-Permanent Income Hypothesis. Theory and Evidence”. In: *The Journal of Political Economy* 86.6, pp. 971–987.
-  Varian, Hal R. (1992). *Microeconomic Analysis*. 3rd ed.
-  – (2014). *Intermediate Microeconomics. With Calculus*. 1st ed.
-  Williamson, Stephen D. (2014). *Macroeconomics*. 5th ed. Pearson.