The consumer's problem(s)

Randall Romero Aguilar, PhD

I Semestre 2019
Last updated: April 22, 2019

Table of contents

1. Introducing the representative consumer

2. Preferences and utility functions

3. Feasible bundles and budget constraints

4. The representative consumer problem

5. Examples

Introduction

- In this course we are interested in
  - being explicit about the role of expectations in the economy; and
  - building macroeconomic models based on microfoundations.
- Last four lectures we studied a modified IS-LM version, incorporating expectations.
- We now turn our attention to microfoundations.
- This presentation is partially based on Williamson (2014, chapter 4) and Jehle and Reny (2001, chapter 1).

Introducing the representative consumer

The representative consumer

- For macroeconomic purposes, it's convenient to suppose that all consumers in the economy are identical.
- In reality, of course, consumers are not identical, but for many macroeconomic issues diversity among consumers is not essential to addressing the economics of the problem at hand, and considering it only clouds our thinking.

The representative consumer (cont'n)

- Identical consumers, in general, behave in identical ways, and so we need only analyze the behavior of one of these consumers: the representative consumer, who acts as a stand-in for all of the consumers in the economy.
- Further, if all consumers are identical, the economy behaves as if there were only one consumer, and it is, therefore, convenient to write down the model as having only a single representative consumer.

Our task

- We show how to represent a consumer's preferences over the available goods in the economy and how to represent the consumer's budget constraint, which tells us what goods are feasible for the consumer to purchase given market prices.
- We then put preferences together with the budget constraint to determine how the consumer behaves given market prices.

The optimization principle

- A fundamental principle that we adhere to here is that consumers optimize: consumer wishes to make himself as well off as possible given the constraints he faces.
- The optimization principle is a very powerful and useful tool in economics, and it helps in analyzing how consumers respond to changes in the environment in which they live.

Available choices

- It proves simplest to analyze consumer choice to suppose that there are two goods that consumers desire.
- Let's call the goods X and Y.
- These "goods" will represent different things in our several models. Examples:
  - two physical goods
  - one consumption good and leisure
  - a consumption good in the present and in the future
  - two financial assets
  - an ice cream in a sunny day vs a rainy day

Preferences and utility functions

Describing the representative consumer preferences

- A key step in determining how the representative consumer makes choices is to capture his preferences over the two goods by a utility function U, written as

$$U(x, y),$$

where x is the quantity of good X, and y is the quantity of good Y.

- We refer to a particular combination of the two goods (x_1, y_1) as a consumption bundle.
- It is useful to think of $U(x_1, y_1)$ as giving the level of happiness, or utility, that the consumer receives from consuming the bundle (x_1, y_1).

Utility functions as rankings for consumption bundles

- The actual level of utility, however, is irrelevant; all that matters for the consumer is what the level of utility is from a given consumption bundle relative to another one.
- This allows the consumer to rank different consumption bundles.
- That is, suppose that there are two different consumption bundles (x_1, y_1) and (x_2, y_2). We say that the consumer...

strictly prefers (x_1, y_1) to (x_2, y_2) if $U(x_1, y_1) > U(x_2, y_2)$
strictly prefers (x_2, y_2) to (x_1, y_1) if $U(x_1, y_1) < U(x_2, y_2)$
is indifferent between the two bundles if $U(x_1, y_1) = U(x_2, y_2)$

Assumptions about preferences

- To use our representation of the consumer's preferences for analyzing macroeconomic issues, we must make some assumptions concerning the form that preferences take.
- These assumptions are useful for making the analysis work, and they are also consistent with how consumers actually behave.
- We assume that the representative consumer's preferences have three properties:
    1. more is preferred to less;
    2. the consumer likes diversity in his or her consumption bundle; and
    3. both goods are normal goods.

1. More is always preferred to less

- A consumer always prefers a consumption bundle that contains more X, more Y, or both.
- This may appear unnatural, because it seems that we can get too much of a good thing.
- This implies that U must be increasing in both x and y, that is:

$$\begin{aligned} x_1 > x_0 \quad &\Leftrightarrow \quad U(x_1, y) \geq U(x_0, y), \quad \forall y \\ y_1 > y_0 \quad &\Leftrightarrow \quad U(x, y_1) \geq U(x, y_0), \quad \forall x \end{aligned}$$

- If U is differentiable, we can express this by

$$U_x \equiv \frac{\partial U}{\partial x} > 0, \qquad\qquad U_y \equiv \frac{\partial U}{\partial y} > 0$$

2. The consumer likes diversity in his bundle

- If the representative consumer is indifferent between two bundles with different combinations of X and Y, a preference for diversity means that any mixture of the two bundles is preferable to either one.
- In terms of the utility function:

$$U(x_0, y_0) = U(x_1, y_1) \Rightarrow \\ U(\alpha x_0 + (1-\alpha)x_1, \alpha y_0 + (1-\alpha)y_1) > U(x_0, y_0)$$

  where $0 < \alpha < 1$.
- If U satisfies this property, it is said to be strictly quasiconcave.

3. Both goods are normal goods

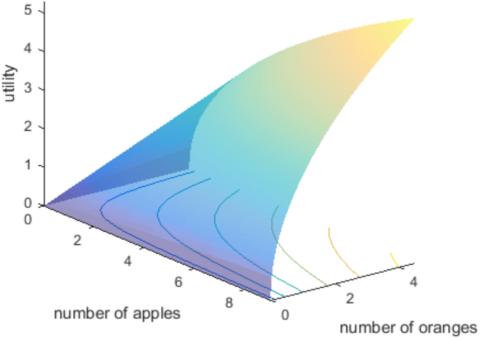
- In some models, we will assume that both goods are normal.
- A good is normal for a consumer if the quantity of the good that he purchases increases when income increases.
- In contrast, a good is inferior for a consumer if he purchases less of that good when income increases.

The utility function

- A consumer's preferences over two goods x and y are defined by the utility function

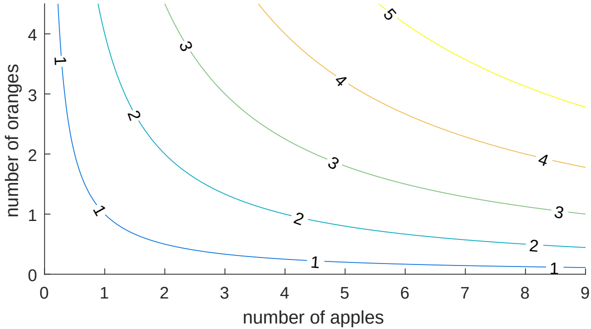
$$U(x, y)$$

- The $U(\,\cdot\,,\,\cdot\,)$ function is increasing in both goods, strictly quasiconcave, and twice differentiable.



Indifference curves

- Consumer's preferences are depicted by a graphical representation of $U(x, y)$, called the indifference map.
- The indifference map is a family of indifference curves.
- An indifference curve connects a set of points, with these points representing consumption bundles, among which the consumer is indifferent



The marginal rate of substitution

- The total derivative for the utility function is:

$$\mathrm{d}U = U_x \; \mathrm{d}x + U_y \; \mathrm{d}y$$

- For an indifference curve $\mathrm{d}U = 0$ (constant utility) the slope is

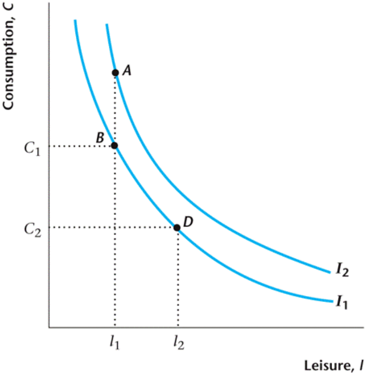
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{dU=0} = -\frac{U_x}{U_y}$$

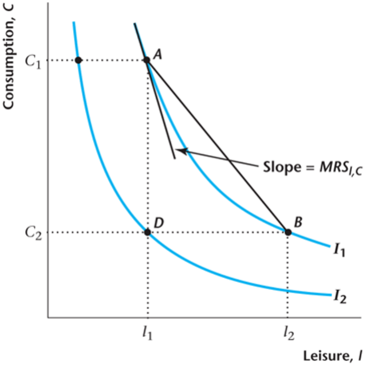
- The marginal rate of substitution of X for Y, denoted $\text{MRS}_{X,Y}$ is the rate at which the consumer is just willing to substitute good X for good Y.
- The $\text{MRS}_{X,Y}$ is equal to the (negative) slope of and indifference curve passing by (x, y).

Properties of an indifference curve

An indifference curve has two key properties:

1. it is downward-sloping (because more is better than less)
2. it is convex (because consumer likes diversity)



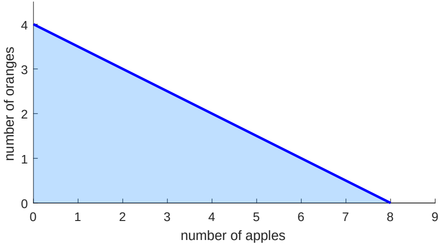


Feasible bundles and budget constraints

The budget constraint

- In order to consume x and y, the consumer must purchase them in the market, at (dollar) prices p_x and p_y.
- We assume that markets are competitive, so the consumer must take prices as given.
- The consumer has M dollars to spend. M does not depend on x or y, but may depend on prices.
  - The consumer cannot spend more than his resources. His budget constraint is

$$p_x x + p_y y \leq M$$



Relative price

- Solving the budget constraint for y we have

$$y = \frac{M}{p_y} - \frac{p_x}{p_y}x$$

- So the slope of the budget constraint is $-\frac{p_x}{p_y}$: the (negative) relative price of X in terms of Y.
- The relative price X in terms of Y represents the number of units of Y that a consumer must forfeit in order to get an additional unit of X.

A barter economy

- In many models we assume an economy without monetary exchange: a barter economy.
- In a barter economy, all trade involves exchanges of goods for goods.
- In such models,
  - we normalize $p_y = 1$ and denote by p the relative price of X in terms of Y.
  - instead of having M units of money, the consumer will have an initial endowment of goods $(\bar{x}, \bar{y})$
- With these assumptions, the budget constraint would be

$$px + y \leq p\bar{x} + \bar{y}$$

The representative consumer problem

The consumers problem

- The consumer's optimization problem is to choose x and y so as to maximize $U(x,y)$ subject to $p_x x + p_y y \leq M$
- The associated Lagrangian is

$$\mathcal{L}(x,y,\lambda) = U(x,y) + \lambda(M - p_x x - p_y y)$$

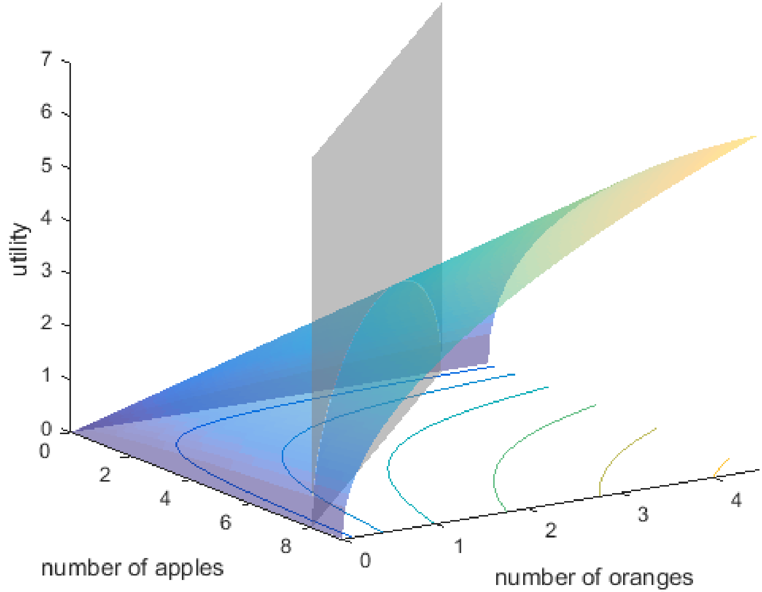
- The first-order conditions are

$$\left.\begin{aligned} U_x(x,y) - \lambda p_x = 0 \\ U_y(x,y) - \lambda p_y = 0 \end{aligned}\right\} \quad \Rightarrow \quad \frac{U_x}{U_y} = \frac{p_x}{p_y}$$

- and the slackness condition

$$\lambda \geq 0, \qquad M - p_x x - p_y y \geq 0, \qquad \lambda(M - p_x x - p_y y) = 0$$

Graphical representation of the consumer's problem, 3D



Don't leave anything on the table

- Suppose that consumer does not spend all his resources:

$$p_x x + p_y y < M$$

- From the slackness conditions, this would imply $\lambda = 0$.
- But from FOCs,

$$U_x = U_y = 0$$

  which contradicts that assumption that consumer is insatiable (marginal utilities are always positive).
- Therefore, it must be the case that $\lambda > 0$ and, because of the slackness condition,

$$p_x x + p_y y = M$$

  that is, the consumer always spends all his resources.
- From now on, we simply assume that the budget constraint is always binding.

Marshallian demand

- Dividing one FOC by the other we get that the solution x^*, y^* must satisfy

$$\frac{U_x(x^*, y^*)}{U_y(x^*, y^*)} = \frac{p_x}{p_y} \qquad \big(\text{MgRS} = \text{relative price}\big)$$

$$p_x x^* + p_y y^* = M \qquad \big(\text{spend all resources}\big)$$

- The solution values will depend on prices and income:

$$x^* = x^*(p_x, p_y, M) \qquad y^* = y^*(p_x, p_y, M)$$

- We refer to these functions as the Marshallian demand functions for X and Y.

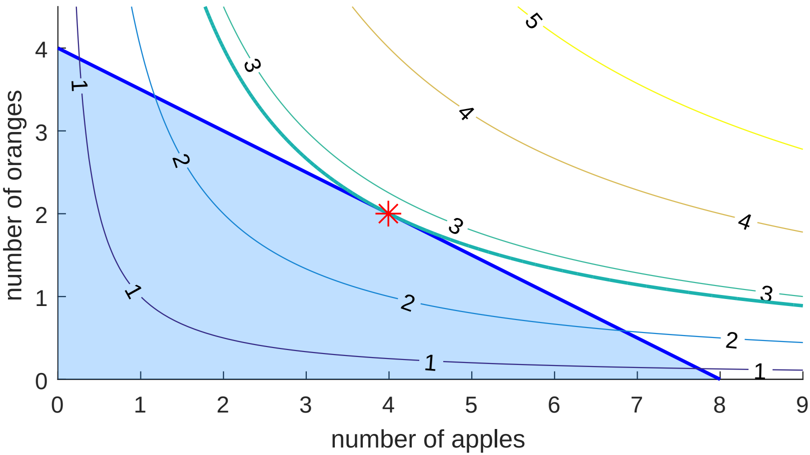
Interpretation of the FOCs

- The "MgRS $=$ relative price condition" can also be written as:

$$\lambda^* = \frac{U_x(x^*, y^*)}{p_x} = \frac{U_y(x^*, y^*)}{p_y}$$

- Say you reduce expenditure on X by \$1, and use it to buy more Y. Then
    - Consumption of X decreases by $\frac{1}{p_x}$, reducing utility by $\frac{U_x}{p_x}$
    - Consumption of Y increases by $\frac{1}{p_y}$, increasing utility by $\frac{U_y}{p_y}$
    - Change in utility is $\frac{U_y}{p_y} - \frac{U_x}{p_x}$. If positive, we could increase U by substituting X with Y. If negative, just substitute Y with X. This would contradict that x^*, y^* was optimum.
- In the optimum, the marginal utility of an extra dollar must be the same for all goods.

Graphical representation of the consumer's problem, 2D



Indirect utility

- $U(x, y)$ is defined over the set of consumption bundles and represents the consumer's preference directly, so we call it the direct utility function
- Given prices and income, the consumer chooses a utility-maximizing bundle (x^*, y^*).
- The level of utility achieved when this bundle is chosen is the highest level permitted by the budget constraint. It changes when prices or income change.
- We define the indirect utility function as

$$\begin{aligned} V(p_x, p_y, M) &= \max_{x,y} \{U(x, y) \text{ s.t. } p_x x + p_y y = M\} \\ &= U\left(x^*(p_x, p_y, M),\ y^*(p_x, p_y, M)\right) \end{aligned}$$

An envelope condition

- Let's say you need to compute the partial derivatives of the Lagrangian wrt income and prices. Then

$$\begin{aligned}
\mathcal{L} &= U(x^*, y^*) + \lambda^*(M - p_x x^* - p_y y^*) \\
\frac{\partial \mathcal{L}}{\partial M} &= \lambda^* + (U_x - \lambda^* p_x)\frac{\partial x^*}{\partial M} + (U_y - \lambda^* p_y)\frac{\partial y^*}{\partial M} &&= \lambda^* \\
\frac{\partial \mathcal{L}}{\partial p_x} &= -\lambda^* x^* + (U_x - \lambda^* p_x)\frac{\partial x^*}{\partial p_x} + (U_y - \lambda^* p_y)\frac{\partial y^*}{\partial p_x} &&= -\lambda^* x^* \\
\frac{\partial \mathcal{L}}{\partial p_y} &= -\lambda^* y^* + (U_x - \lambda^* p_x)\frac{\partial x^*}{\partial p_y} + (U_y - \lambda^* p_y)\frac{\partial y^*}{\partial p_y} &&= -\lambda^* y^*
\end{aligned}$$

- That is, we can derive the Lagrangian wrt the parameters as if x^* and y^* did not depend on those parameters!

The shadow price of income

- Remember that at optimum $\lambda^* = \frac{U_x}{p_x} = \frac{U_y}{p_y}$
- Let's obtain the marginal value of an extra unit of income:

$$\begin{aligned}
\frac{\partial V(p_x, p_y, M)}{\partial M} &= \frac{\partial U(x^*,\ y^*)}{\partial M} \\
&= U_x \frac{\partial x^*}{\partial M} + U_y \frac{\partial y^*}{\partial M} \\
&= \lambda^* p_x \frac{\partial x^*}{\partial M} + \lambda^* p_y \frac{\partial y^*}{\partial M} \\
&= \lambda^* \left(p_x \frac{\partial x^*}{\partial M} + p_y \frac{\partial y^*}{\partial M} \right) \\
&= \lambda^* \qquad \qquad \text{(because of budget constraint)}
\end{aligned}$$

- That is, λ^* represents the marginal utility of an extra unit of income.

Roy's identity

- It is easy to obtain the *indirect utility function* from the *Marshallian demand functions*: we just substitute the Marshallian demands into the *direct* utility function.
- To obtain the Marshallian demand functions from the indirect utility function, we use Roy's identity:

$$x^* = -\frac{\frac{\partial V}{\partial p_x}}{\frac{\partial V}{\partial M}} \qquad\qquad y^* = -\frac{\frac{\partial V}{\partial p_y}}{\frac{\partial V}{\partial M}}$$

where we assume that V is differentiable and $\frac{\partial V}{\partial M} \neq 0$

A strictly positive monotone transformation

- Suppose that you rescale the utility function using a strictly increasing function f, where $f'(\cdot) > 0$
- That is, you use

$$W(x, y) \equiv f(U(x, y))$$

- Notice that the MRS is just the same as before:

$$\frac{W_x}{W_y} = \frac{f'(U)U_x}{f'(U)U_y} = \frac{U_x}{U_y}$$

- So the optimal consumption bundle will be the same.
- Utility function has ordinal meaning, but no cardinal meaning.
- Warning: This result does not apply to choice problems with uncertainty.

The consumers problem with non-negativity constraints

- Now the consumer's problem is to choose x and y so as to maximize $U(x, y)$ subject to $p_x x + p_y y = M$ and $x \geq 0$, $y \geq 0$.
- The associated Lagrangian is

$$L(x, y, \lambda) = U(x, y) + \lambda(M - p_x x - p_y y) + \mu_1 x + \mu_2 y$$

- The first-order conditions are

$$\begin{aligned} U_x(x, y) - \lambda p_x + \mu_1 &= 0 \\ U_y(x, y) - \lambda p_y + \mu_2 &= 0 \\ p_x x + p_y y &= M \end{aligned}$$

- and the slackness conditions

$$\begin{array}{ccc} \mu_1 \geq 0 & x \geq 0 & \mu_1 x = 0 \\ \mu_2 \geq 0 & y \geq 0 & \mu_2 y = 0 \end{array}$$

- Notice that we can solve for μ_1 and μ_2 in the FOCs.

$$\begin{aligned} U_x(x,y) - \lambda p_x &= -\mu_1 \\ U_y(x,y) - \lambda p_y &= -\mu_2 \end{aligned}$$

- So a solution must satisfy

$$\begin{aligned} U_x(x,y) - \lambda p_x &\leq 0 & x &\geq 0 & x\left(U_x(x,y) - \lambda p_x\right) &= 0 \\ U_y(x,y) - \lambda p_y &\leq 0 & y &\geq 0 & y\left(U_y(x,y) - \lambda p_y\right) &= 0 \\ p_x x + p_y y &= M \end{aligned}$$

Ruling out corner solutions

- From the budget constraint, we know that x and y cannot be simultaneously zero (assuming $M > 0$)
- Assume that $x = 0$. Notice that the solution requires that

$$U_x(x, y) \leq \lambda p_x < \infty$$

- If the marginal utility of x as x approaches zero is infinite:

$$\lim_{x \to 0} U_x(x, y) = \infty$$

  we would get a contradiction.
- Therefore

$$\lim_{x \to 0} U_x(x, y) = \infty \to x > 0 \Rightarrow U_x(x, y) - \lambda p_x = 0$$
$$\lim_{y \to 0} U_y(x, y) = \infty \to y > 0 \Rightarrow U_y(x, y) - \lambda p_y = 0$$

Summary

For the consumer's problem of choosing x and y so as to maximize $U(x, y)$ subject to $p_x x + p_y y \leq M$ and $x \geq 0$, $y \geq 0$:

- If utility is strictly increasing in its arguments, then in the solution the budget constraint is binding.
- If the marginal utilities tend to infinity as any of its arguments goes to zero, then in the solution the non-negativity constraints are not binding.

Examples

Example 1:

A Cobb-Douglas utility function

- The Cobb-Douglas utility function is

$$U(x, y) = x^{\theta} y^{1-\theta} \qquad \text{(where } 0 < \theta < 1\text{)}$$

- The marginal utilities of x and y are

$$\frac{\partial U}{\partial x} = \theta \left(\frac{y}{x}\right)^{1-\theta} \qquad \frac{\partial U}{\partial y} = (1-\theta)\left(\frac{x}{y}\right)^{\theta}$$

- If $x > 0$ and $y > 0$, we have

$$\lim_{x \to 0} U_x = \infty \qquad \lim_{y \to 0} U_y = \infty \qquad U_x > 0 \qquad U_y > 0$$

- Therefore, if the utility is Cobb-Douglas, we know that
  - there will be no corner solution ($x^* > 0$ and $y^* > 0$).
  - the consumer will spend all his resources ($p_x x^* + p_y y^* = M$)

- The optimal bundle must satisfy

$$\frac{\theta\left(\frac{y}{x}\right)^{1-\theta}}{(1-\theta)\left(\frac{x}{y}\right)^{\theta}} = \frac{p_x}{p_y} \qquad \text{(MgRS = relative price)}$$

$$p_x x^* + p_y y^* = M \qquad \text{(budget constraint)}$$

- The Marshallian demands are:

$$x^*(p_x, p_y, M) = \frac{\theta M}{p_x} \qquad y^*(p_x, p_y, M) = \frac{(1-\theta)M}{p_y}$$

- The indirect utility function is

$$V(p_x, p_y, M) = \left(\frac{\theta}{p_x}\right)^{\theta}\left(\frac{1-\theta}{p_y}\right)^{1-\theta} M$$

Example 2:

A quasi-linear utility function

Consider the quasi-linear utility function

$$U(x, y) = \ln x + y$$

Find the optimal allocation for this problem.

- Given that $U_x = \frac{1}{x} > 0$ and $U_y = 1 > 0$, we know that the consumer spends all his income.
- In this problem, y is the numeraire, so $p_y = 1$. Let $p_x = p$.
- Lagrangian is

$$L = \ln x + y + \lambda(M - px - y)$$

- Optimality conditions:

$$\left.\begin{aligned} \frac{1}{x} - \lambda p &= 0 \\ 1 - \lambda &= 0 \\ px + y &= M \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} \lambda &= 1 \\ px &= 1 \\ y &= M - 1 \end{aligned}$$

- Marshallian demand:

$$x^*(p, M) = \frac{1}{p} \qquad y^*(p, M) = M - 1$$

- Notice that y^* is negative if $M < 1$.
- Indirect utility:

$$V(p, M) = M - 1 - \ln p$$

Example 3:

A quasi-linear utility function, with non-negativity constraints

Now assume that neither x nor y can be negative.

- Since $U_x = \frac{1}{x} \to \infty$ as $x \to 0$, we know that $x > 0$.
- But $U_y = 1$, so y could be zero.
- Lagrangian is

$$L = \ln x + y + \lambda(M - px - y) + \mu y$$

- Optimality conditions:

$$\begin{aligned} &\frac{1}{x} - \lambda p = 0 \\ &1 - \lambda \leq 0 \qquad y \geq 0 \qquad (1-\lambda)\, y = 0 \\ &px + y = M \end{aligned}$$

- We need to consider two cases:

$$\lambda = 1 \qquad \text{or} \qquad y = 0$$

Case 1: $\lambda = 1$

From FOC wrt x, we have $\lambda^{-1} = px$. So, in this case, $px = 1$. Substitute in the budget constraint to get

$$y^* = M - 1 \qquad \text{and} \qquad x^* = \frac{1}{p}$$

But we need to make sure that $y \geq 0$, so we require that $M \geq 1$.

Case 2: $y = 0$

From the budget constraint, $px = M$, so a candidate solution is

$$y^* = 0 \qquad \text{and} \qquad x^* = \frac{M}{p}$$

But we need to make sure that $1 - \lambda \leq 0$. Again, we know that $\lambda^{-1} = px = M$, so this condition is equivalent to

$$1 - \frac{1}{M} \leq 0 \quad \Leftrightarrow \quad M \leq 1$$

Taking the two cases together, we have the solution:

- Marshallian demand

$$x^* = \frac{\min\{M, 1\}}{p}, \qquad y^* = \max\{M - 1, 0\}$$

- Indirect utility

$$\begin{aligned} V(p, M) &= \max\{M, 1\} - 1 + \ln \min\{M, 1\} - \ln p \\ &= \begin{cases} \ln M - \ln p, & \text{if } M \leq 1 \\ M - 1 - \ln p, & \text{otherwise} \end{cases} \end{aligned}$$

- Notice that V is continuous and differentiable at $M = 1$.

Example 4:

Many goods, CES utility

- Now assume that there are $n+1$ goods available to the consumer.
- Let p_i and x_i be the price and quantity consumed of good X_i, for $i = 0, 1, \dots, n$.
- Assume a CES utility function, with $\alpha_i > 0 \; \forall i$

$$U(x_0, x_1, \dots, x_n) = \left(\sum_{i=0}^{n} \alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} = (\alpha_0 x_0^{\rho} + \alpha_1 x_1^{\rho} + \cdots + \alpha_n x_n^{\rho})^{\frac{1}{\rho}}$$

- Let X_0 be the numeraire ($p_0 = 1$) and let $\alpha_0 = 1$.
- Budget constraint is

$$\sum_{i=0}^{n} p_i x_i = p_0 x_0 + p_1 x_1 + \cdots + p_n x_n = M$$

- Lagrangian is

$$\mathcal{L} = \left(\sum_{i=0}^{n} \alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} + \lambda\left(M - \sum_{i=0}^{n} p_i x_i\right)$$

- For this problem, it is convenient to replace the utility function using the $f(U) = U^{\rho}$ transformation:

$$\mathcal{L} = \sum_{i=0}^{n} \alpha_i x_i^{\rho} + \lambda\left(M - \sum_{i=0}^{n} p_i x_i\right)$$

$$\mathcal{L} = \sum_{i=0}^{n} \left(\alpha_i x_i^{\rho} - \lambda p_i x_i\right) + \lambda M$$

- First order conditions

$$\rho \alpha_i x_i^{\rho-1} - \lambda p_i = 0 \quad \Rightarrow \quad \lambda = \frac{\rho \alpha_i x_i^{\rho-1}}{p_i} \qquad \forall i = 0, \dots, n$$

- that is, the marginal utility per dollar must be the same for all goods.

- Let's write conditions in terms of numeraire X_0 condition, $i = 1, \dots, n:$

$$\frac{\rho \alpha_i x_i^{\rho-1}}{p_i} = \frac{\rho \alpha_0 x_0^{\rho-1}}{p_0} \quad \Rightarrow \quad x_i = \alpha_i^\sigma p_i^{-\sigma} x_0 \quad \left(\text{with } \sigma = \tfrac{1}{1-\rho}\right)$$

- Substitute in budget constraint to get:

$$M = \sum_{i=0}^{n} p_i x_i = x_0 + \sum_{i=1}^{n} \alpha_i^\sigma p_i^{1-\sigma} x_0 \quad \Rightarrow x_0^* = \frac{M}{1 + \sum_{i=1}^{n} \alpha_i^\sigma p_i^{1-\sigma}}$$

- Therefore, the Marshallian demands satisfy:

$$p_k x_k^* = \frac{\alpha_k^\sigma p_k^{1-\sigma}}{\sum_{i=0}^{n} \alpha_i^\sigma p_i^{1-\sigma}} M \qquad \forall k = 0, \dots, n$$

- With CES utility, the income share spent on each good depends on the preference parameters and on the prices of all goods.

- Let the price index P be defined by

$$P \equiv \left(\sum_{i=0}^{n} \alpha_i^{\sigma} p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

- Marshallian demand is then

$$x_k^* = \frac{\alpha_k^{\sigma} p_k^{-\sigma}}{P^{1-\sigma}} M = \left(\frac{\alpha_k}{\frac{p_k}{P}}\right)^{\sigma} \frac{M}{P} \qquad \forall k = 0, \dots, n$$

- We can interpret $\frac{M}{P}$ as a measure of real consumption.
- For each good, demand is a share of real consumption. The share depends on:
  - α_k, the weight of the good in the utility function
  - σ, the elasticity of substitution between goods, and
  - p_x/P, the price of the good relative to the price index.

To get the indirect utility function, use the demand functions

$$
\begin{aligned}
x_k^* &= \alpha_k^\sigma p_k^{-\sigma} M P^{\sigma-1} \\
\alpha_k x_k^{*\rho} &= \alpha_k^{1+\sigma\rho} p_k^{-\sigma\rho} (M P^{\sigma-1})^\rho \\
&= \alpha_k^\sigma p_k^{1-\sigma} (M P^{\sigma-1})^\rho \\
\left(\sum_{i=0}^n \alpha_i x_i^{*\rho}\right)^{\frac{1}{\rho}} &= \left(\sum_{i=0}^n \alpha_k^\sigma p_k^{1-\sigma} (M P^{\sigma-1})^\rho\right)^{\frac{1}{\rho}} \\
V\left(p_0, \dots, p_n, M\right) &= M P^{\sigma-1} \left(\sum_{i=0}^n \alpha_k^\sigma p_k^{1-\sigma}\right)^{\frac{1}{\rho}} \\
&= M P^{\sigma-1} P^{\frac{1-\sigma}{\rho}} = \frac{M}{P}
\end{aligned}
$$

We can interpret V as a measure of real consumption.

- **Case: Cobb-Douglas utility function** is the special case where $\rho = 0$ (equivalently $\sigma = 1$):

$$U(x_0, x_1, \dots, x_n) = \prod_{i=0}^{n} x_i^{\alpha_i} = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

- In this case, the Marshallian demands satisfy:

$$p_i x_i^* = \frac{\alpha_i}{\sum_{i=0}^{n} \alpha_i} M \qquad \forall i = 0, \dots, n$$

- With Cobb-Douglas utility, the share of income spent on each of the goods does not depend on any price.

References I

- Jehle, Geoffrey A. and Philip J. Reny (2001). *Advanced Microeconomic Theory*. 2nd ed. Addison Wesley. ISBN: 978-0321079169.
- Williamson, Stephen D. (2014). *Macroeconomics*. 5th ed. Pearson.