# Firm's investment

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# Objective

- In this lecture we determine how the value of the firm is affected by its decisions on investment and production, capital structure, and dividend policy.
- We also apply dynamic programing to determine the optimal investment policy for a risk-neutral firm facing quadratic capital-adjustment costs.

# The value of a firm

# Value of the firm

| Balance sheet | |
| --- | --- |
| * productive capital | * debt (net) |
| | * equity |

- In practice, easier to measure the liabilities:

$$\text{value of the firm} = \text{debt} + \text{equity}$$

# Questions

How does the value of the firm depend upon

- investment and production?
- capital structure?
- dividend policy?

# Cash flows constraint

## Cash flow

| | |
|---|---|
| Sales of output | $P_{Y_t} Y_t$ |
| - wage cost | $-w_t N t$ |
| - investment | $-P_{I_t} I_t$ |
| - adjustment cost | $-P_{I_t} G\left(I_t, K_t\right)$ |
| = net cash flow from production | $= X_t$ |
| - dividend | $-d_t S_t$ |
| + new equity | $+P_{S_t}\left(S_{t+1} - S_t\right)$ |
| + bond emission | $+P_{B_t} B_{t+1}$ |
| - bond repayment | $-B_t$ |
| = net cash flow | $= 0$ |

# The firm: main equations

- Production function

$$Y_t = F\left(N_t, K_t; A_t\right)$$

where $N$ is labor, $K$ is capital, and $A$ is productivity.

- Capital accumulation

$$K_t = \lambda K_{t-1} + I_t$$

where $\lambda$ equals 1 minus the depreciation rate.

- Cash flow constraint:

$$X_t - d_t S_t + P_{S_t}\left(S_{t+1} - S_t\right) + P_{B_t} B_{t+1} - B_t = 0$$

- which implies that :

$$X_t + P_{S_t} S_{t+1} + P_{B_t} B_{t+1} = \left(d_t + P_{S_t}\right) S_t + B_t$$

# Asset pricing

- Remember form Lecture 12 on asset pricing that:

$$1 = \mathbb{E}_t \left[ \frac{y_{it+1} + P_{it+1}}{P_{it}} \times \frac{\beta u'(C_{t+1})}{u'(C_t)} \right] = \mathbb{E}_t [R_{it} M_t]$$

- Applying this result to the firm equity:

$$1 = \mathbb{E}_t \left[ \frac{d_{t+1} + P_{St+1}}{P_{St}} M_t \right] \quad \Rightarrow P_{St} = \mathbb{E}_t \left[ \left( d_{t+1} + P_{St+1} \right) M_t \right]$$

- whereas applying it to the firm debt (assuming risk-free)

$$1 = \mathbb{E}_t \left[ \frac{1}{P_{Bt}} M_t \right] \quad \Rightarrow P_{Bt} = \mathbb{E}_t \left[ 1 M_t \right]$$

# A recursive formula for the value of the firm

- The value of the firm (before dividends) is

$$
\begin{aligned}
V_t \equiv\ & \left(d_t + P_{S_t}\right) S_t + B_t \\
& = X_t + P_{S_t} S_{t+1} + P_{B_t} B_{t+1}
\end{aligned}
$$

- Since

$$
\begin{aligned}
P_{St} &= \mathbb{E}_t \left[\left(d_{t+1} + P_{St+1}\right) M_t\right] \\
P_{Bt} &= \mathbb{E}_t \left[M_t\right]
\end{aligned}
$$

- we get

$$
\begin{aligned}
V_t &= X_t + S_{t+1}\, \mathbb{E}_t \left[\left(d_{t+1} + P_{St+1}\right) M_t\right] + B_{t+1}\, \mathbb{E}_t \left[M_t\right] \\
&= X_t + \mathbb{E}_t \left\{M_t \left[\left(d_{t+1} + P_{St+1}\right) S_{t+1} + B_{t+1}\right]\right\} \\
&= X_t + \mathbb{E}_t \left\{M_t V_{t+1}\right\}
\end{aligned}
$$

# Value of the firm depends on net cash flow from production

Substitution of $V_{t+1}$ on the $V_t$ results in:

$$\begin{aligned}
V_t &= X_t + \mathbb{E}_t\left\{M_t V_{t+1}\right\} \\
&= X_t + \mathbb{E}_t\left\{M_t\left[X_{t+1} + \mathbb{E}_{t+1}\left(M_{t+1}V_{t+2}\right)\right]\right\} \\
&= X_t + \mathbb{E}_t\left[M_t X_{t+1}\right] + \mathbb{E}_t\left\{M_t\,\mathbb{E}_{t+1}\left(M_{t+1}V_{t+2}\right)\right\} \\
&= X_t + \mathbb{E}_t\left[M_t X_{t+1}\right] + \mathbb{E}_t\left\{M_t M_{t+1} V_{t+2}\right\}
\end{aligned}$$

Last step follows from the Law of Iterated Expectations.

# Value of the firm depends on net cash flow from production

- If we keep iterating we get:

$$V_t = \mathbb{E}_t \left[ \sum_{s=t}^{\infty} M_{t,s} X_s \right] + \lim_{s \to \infty} \mathbb{E}_t \left( M_{t,s} V_s \right)$$

  where

$$M_{t,s} \equiv M_t M_{t+1} \ldots M_{s-1} = \beta^{s-t} \frac{u'(c_s)}{u'(c_t)}$$

  is the MRS of consumption between periods $s$ and $t$.
- Because $0 < \beta < 1$, it follows that $\lim_{s \to \infty} M_{t,s} = 0$
- Therefore, the value of the firm equals the present value of the net cash flow from production:

$$V_t = \mathbb{E}_t \left[ \sum_{s=t}^{\infty} M_{t,s} X_s \right]$$

# Modigliani-Miller theory

**Firm value**

$$V_t = \mathbb{E}_t \left[ \sum_{s=t}^{\infty} M_{t,s} X_s \right]$$

- The value of the firm equals the present discounted value of its net cash flow from production $X_t$.
- The value of the firm does not depend upon:
  - the debt-equity ratio
  - the dividend policy

## Modigliani-Miller theory (2)

| Firm value | $$V_t = \mathbb{E}_t\left[\sum_{s=t}^{\infty} M_{t,s} X_s\right]$$ |
|---|---|

Notice that these results depend on some underlying assumptions:

- no friction in capital market:
  - no private information
  - no difficulty of contract enforcement
- no taxes

# Investment by a risk-neutral firm

# A risk-neutral firm

- For a risk-neutral firm, $M_{t,s} = \beta^{s-t}$ and

$$V_t = X_t + \beta\,\mathbb{E}_t\left(V_{t+1}\right)$$

- The Bellman equation is:

$$V_t\left(\lambda K_{t-1}\right) = \max_{N_t, I_t} \left\{ P_{Y_t} Y_t - w_t N_t - P_{I_t}\left[I_t + G\left(I_t, K_t\right)\right] + \right. \\ \left. + \beta\,\mathbb{E}_t\, V_t\left(\lambda K_t\right) \right\}$$

subject to

$$K_t = \lambda K_{t-1} + I_t \qquad \text{and} \qquad Y_t = F\left(N_t, K_t, A_t\right)$$

# Solving the problem

- The FOCs are:

$$0 = \underbrace{P_{Y_t} F_{N_t}}_{\text{Marginal product labor}} - w_t$$

$$0 = \underbrace{P_{Y_t} F_{K_t} - P_{I_t} G_{K_t}}_{\text{Marginal product capital}} - \underbrace{P_{I_t}\left(1 + G_{I_t}\right)}_{\text{Mg cost Investm.}} + \beta\lambda\,\mathbb{E}_t\left(\frac{\partial V_{t+1}}{\partial(\lambda K_t)}\right)$$

- where

$$F_N \equiv \frac{\partial F}{\partial N}, \quad F_K \equiv \frac{\partial F}{\partial K}, \quad G_I \equiv \frac{\partial G}{\partial I}, \quad G_K \equiv \frac{\partial G}{\partial K}$$

- From now on we denote the marginal product of capital by $\text{MPK}_t$

# The marginal value of capital

- From the envelope condition we find the marginal value of capital $q_t^*$:

$$
\begin{aligned}
q_t^* \equiv \frac{\partial V_t}{\partial(\lambda K_{t-1})} &= P_{Y_t} F_{K_t} - P_{I_t} G_{K_t} + \beta\lambda\,\mathbb{E}_t\left(\frac{\partial V_{t+1}}{\partial(\lambda K_t)}\right) \\
&= \text{MPK}_t + \beta\lambda\,\mathbb{E}_t\left(\frac{\partial V_{t+1}}{\partial(\lambda K_t)}\right) \\
&= \underbrace{P_{I_t}\left(1 + G_{I_t}\right)}_{\text{Mg cost Investm.}}
\end{aligned}
$$

- the last equality follows from the second FOC

# Iterating on the envelope condition

$$
\begin{aligned}
q_t^* \equiv \frac{\partial V_t}{\partial(\lambda K_{t-1})} &= \mathsf{MPK}_t + \beta\lambda\,\mathbb{E}_t\left(\frac{\partial V_{t+1}}{\partial(\lambda K_t)}\right) \\
&= \mathsf{MPK}_t + \beta\lambda\,\mathbb{E}_t\left(\mathsf{MPK}_{t+1} + \beta\lambda\,\mathbb{E}_{t+1}\left(\frac{\partial V_{t+2}}{\partial(\lambda K_{t+1})}\right)\right) \\
&= \qquad \vdots \\
&= \mathbb{E}_t\left[\sum_{s=t}^{\infty}(\beta\lambda)^{s-t}\mathsf{MPK}_s\right]
\end{aligned}
$$

# Gathering our results

- So far we have

$$
\begin{aligned}
q_t^* &\equiv \frac{\partial V_t}{\partial(\lambda K_{t-1})} \\
&= \mathbb{E}_t\left[\sum_{s=t}^{\infty}(\beta\lambda)^{s-t}\mathsf{MPK}_s\right] \\
&= P_{I_t}\left(1+G_{I_t}\right)
\end{aligned}
$$

- Then, at the optimum investment, the marginal value of capital corresponds to the present discounted value of the marginal product of capital, and it must equal the marginal cost of investment.

# Quadratic cost of investment adjustment

- Suppose that

$$G(I, K; \epsilon) = \frac{b}{2} \left( \frac{I}{K} - a - \epsilon \right)^2 K$$

- Then

$$G_{I_t}(I_t, K_t; \epsilon_t) = b \left( \frac{I_t}{K_t} - a - \epsilon \right)$$

- and optimal investment requires

$$q_t^* = P_{I_t} \left[ 1 + b \left( \frac{I_t}{K_t} - a - \epsilon_t \right) \right]$$

# Optimal investment

| Optimal Investment | $$\frac{I_t}{K_t} = a + \frac{1}{b}\left(\frac{q_t^*}{P_{I_t}} - 1\right) + \epsilon_t$$ |
|---|---|

- This says that it is optimal to increase the growth rate of capital if the increase in firm value induced by the additional capital is higher than the market price of that capital.