# Markov processes

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# Introduction

- Markov processes are an indispensable ingredient of DSGE models.
- They preserve the recursive structure that these models inherit from their deterministic relatives.
- In this lecture we review a few results about these processes that we will need repeatedly in the modeling of business cycles.

# Stochastic process

# Stochastic Process

A stochastic process is a time sequence of random variables $\{Y_t\}_{t=-\infty}^{\infty}$.

Two types of processes:

**Continuous** if realizations are taken from an interval of the real line $Y_t \in [a, b] \subseteq \mathbb{R}$.

**Discrete** if there is a countable number of realizations $Y_t \in \{y_1, y_2, \ldots, y_n\}$.

# i.i.d. Stochastic Process

- The elements of a stochastic process are identically and independently distributed (iid for short), if the probability distribution is the same for each member of the process $Z_t$ and independent of the realizations of other members of the process.
- In this case

$$\mathbb{P}[Y_1 = y_1, Y_2 = y_2, \ldots, Y_T = y_T] = \\ \mathbb{P}(Y_1 = y_1) \times \mathbb{P}(Y_2 = y_2) \times \cdots \times \mathbb{P}(Y_T = y_T)$$

# Unconditional moments

- Unconditional cumulative distribution function

$$F_{Y_t}(y) = \mathbb{P}[Y_t \leq y]$$

- Unconditional expectation (mean)

$$\mu_t \equiv \mathbb{E}(Y_t) = \int_{-\infty}^{\infty} y \, \mathrm{d}F_{Y_t}(y)$$

- Unconditional variance

$$\gamma_{0t} \equiv \mathbb{E}(Y_t - \mu_t)^2 = \int_{-\infty}^{\infty} (y - \mu_t)^2 \, \mathrm{d}F_{Y_t}(y)$$

- Autocovariance

$$\gamma_{jt} \equiv \mathbb{E}(Y_t - \mu_t)(Y_{t-j} - \mu_{t-j})$$

If neither the mean $\mu_t$ nor the autocovariances $\gamma_{jt}$ depend on the date $t$, then the process for $Z_t$ is said to be covariance-stationary or weakly stationary:

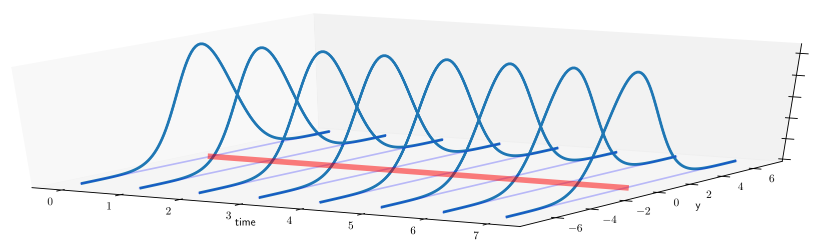
$$\begin{aligned} \mathbb{E}\left(Y_t\right) &= \mu && \text{for all } t \\ \mathbb{E}\left(Y_t - \mu\right)\left(Y_{t-j} - \mu\right) &= \gamma_j && \text{for all } t \text{ and any } j \end{aligned}$$

Example 1:

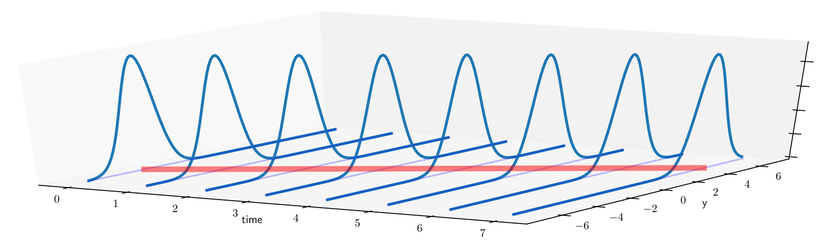
# Stationary and nonstationary processes

Suppose $Y_t$ is a stochastic process such that $Y_t \sim N(\mu_t, \sigma_t^2)$

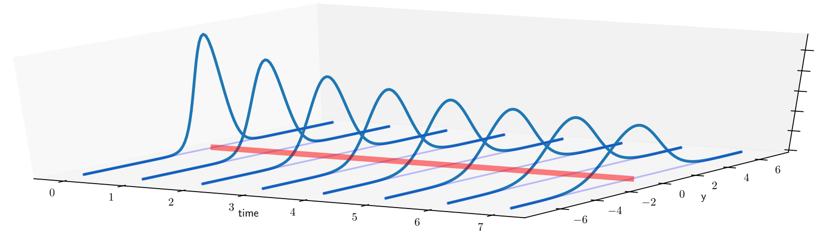
Stationary because $\mu_t$ and $\sigma_t^2$ are constant.



Nonstationary because $\mu_t$ is changing over time.




Nonstationary because $\sigma_t^2$ is changing over time.



# White noise

- The basic building block for the processes considered in this lecture is a sequence $\{\epsilon_t\}$ whose elements have mean zero and variance $\sigma^2$,

$$\mathbb{E}\left(\epsilon_t\right) = 0 \qquad \text{(zero mean)}$$
$$\mathbb{E}\left(\epsilon_t^2\right) = \sigma^2 \qquad \text{(constant variance)}$$
$$\mathbb{E}\left(\epsilon_t \epsilon_\tau\right) = 0 \quad \text{for } t \neq \tau \qquad \text{(uncorrelated terms)}$$

- If the terms are normally distributed

$$\epsilon_t \sim N(0, \sigma^2)$$

then we have the Gaussian white noise process.

# The first-order autoregressive process

# Definition of a AR(1) process

- A first-order autoregression, denoted AR(1), satisfies the following difference equation:

$$Y_t = c + \phi Y_{t-1} + \epsilon_t$$

  where $\{\epsilon_t\}$ is a white noise sequence.
- It is stationary if and only if $|\phi| < 1$.
- In what follows, we assume the process is stationary.

# MA($\infty$) representation of a AR(1) process

- If the AR(1) process is stationary, it can be written

$$Y_t = \frac{c}{1-\phi} + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \phi^3\epsilon_{t-3} + \dots$$

# Conditional versus unconditional mean

- The conditional mean given the previous observation is

$$\mathbb{E}[Y_t \mid Y_{t-1}] = c + \phi Y_{t-1}$$

- The unconditional mean is

$$\mu \equiv \mathbb{E}[Y_t] = \frac{c}{1-\phi}$$

- Since $c = (1-\phi)\mu$, the AR(1) process can be written as deviations from 'equilibrium'

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \epsilon_t$$

# Impulse-response

- Starting with $Y_{t-1}$, the value of $Y_{t+s}$ will be

$$Y_{t+s}-\mu = \phi^{s+1}(Y_{t-1}-\mu)+\phi^{s}\epsilon_t+\phi^{s-1}\epsilon_{t+1}+\cdots+\phi\epsilon_{t+s-1}+\epsilon_{t+s}$$

- Suppose that starting in 'equilibrium' ($Y_{t-1} - \mu = 0$) there is a time-$t$ transitory shock ($\epsilon_t = \nu$) but no more shocks thereafter ($\epsilon_{t+1} = \cdots = \epsilon_{t+s} = 0$). Then

$$Y_{t+s} - \mu = \phi^{s}\nu$$

- This is known as an impulse-response function.
- Notice that the process will return to equilibrium as long as $|\phi| < 1$.

# Conditional versus unconditional variance

- The conditional variance given the previous observation is

$$\mathrm{Var}[Y_t \,|\, Y_{t-1}] = \mathrm{Var}[c + \phi Y_{t-1} + \epsilon_t \,|\, Y_{t-1}] = \sigma^2$$

- The unconditional mean is

$$\gamma_0 \equiv \mathrm{Var}[Y_t] = \frac{\sigma^2}{1 - \phi^2}$$

- Notice that $\gamma_0 > \mathrm{Var}[Y_t \,|\, Y_{t-1}]$

# Autocovariance and autocorrelation

- The autocovariance is given by

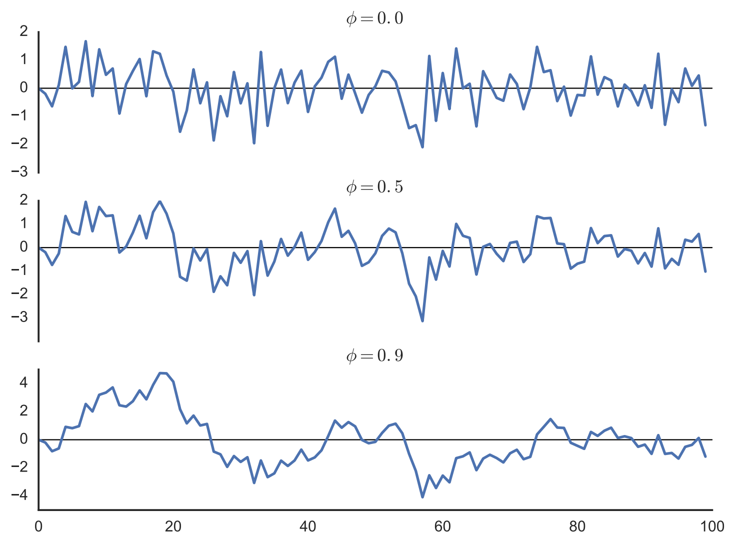
$$\gamma_j = \phi^j \gamma_0 \qquad (j = 1, 2, \ldots)$$

- The autocovariance is given by

$$\rho_j = \phi^j \qquad (j = 1, 2, \ldots)$$

Example 2:

# Realizations of an AR(1) process




The three processes are build from the same white noise realization. Notice how process becomes more persistent as $\phi$ approaches 1.

# Markov chains

# Markov property

A stochastic process $\{Z_t\}_{t=0}^{\infty}$ has the Markov property if for all $k \geq 1$ and all $t$,

$$\mathbb{P}[Z_{t+1} \mid Z_t, Z_{t-1}, \ldots, Z_{t-k}] = \mathbb{P}[Z_{t+1} \mid Z_t]$$

That is, the the probability distribution of $Z_{t+1}$ only depends upon the realization of $Z_t$.

Example 3:

# AR(1) process

- The AR(1) process is a Markov process:

$$Z_{t+1} = (1-\rho)\bar{Z} + \rho Z_t + \epsilon_{t+1}$$

  where $\rho \in [0, 1)$, and $\epsilon_{t+1} \sim \text{iid} N(0, \sigma^2)$ is a white noise process.

- Given $Z_t$, next period's variable $Z_{t+1}$ is normally distributed with:

$$\begin{aligned} \textbf{mean: } & \mathbb{E}(Z_{t+1} \mid Z_t) = (1-\rho)\bar{Z} + \rho Z_t \\ \textbf{variance: } & \mathrm{Var}(Z_{t+1} \mid Z_t) = \sigma^2 \end{aligned}$$

# Markov Chains

Markov chains are discrete valued Markov processes. They are characterized by three objects:

1. The $n$ different realizations of $Z_t$, represented by the column vector $z = [z_1, z_2, \ldots, z_n]'$.
2. The probability distribution of the initial date $t = 0$, $\pi_0 = [\pi_{01}, \pi_{02}, \ldots, \pi_{0n}]'$, where $\pi_{0i} = \mathbb{P}[Z_0 = z_i]$.
3. The transition matrix $P = (p_{ij})$, where $p_{ij} = \mathbb{P}[Z_{t+1} = z_j \mid Z_t = z_i]$, representing the dynamics of the process.

Notice that

- $p_{ij} \geq 0$ and $\sum_{j=1}^{n} p_{ij} = 1$.
- $\pi_{0i} \geq 0$ and $\sum_{i=1}^{n} \pi_{0i} = 1$.

Example 4:

# Unemployment

A worker can either be employed or unemployed:

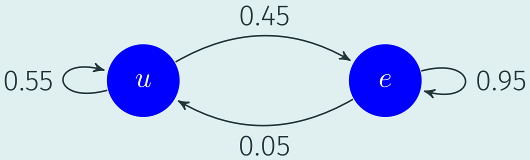
- If unemployed, she will get a job with probability $p = 45\%$
- If employed, she will lose her job with probability $q = 5\%$

The worker is employed at $t = 0$. Then the Markov chain is:

**outcomes** {unemployed, employed} or $z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**initial probability** $\pi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**transition probability** $P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \\ 0.05 & 0.95 \end{bmatrix}$



Example 5:

# Credit ratings

Transition of the credit ratings from one year to the next:

| | AAA | AA | A | BBB | BB | B | CCC | D | N.R. |
|---|---|---|---|---|---|---|---|---|---|
| AAA | 90.34 | 5.62 | 0.39 | 0.08 | 0.03 | 0 | 0 | 0 | 3.5 |
| AA | 0.64 | 88.78 | 6.72 | 0.47 | 0.06 | 0.09 | 0.02 | 0.01 | 3.21 |
| A | 0.07 | 2.16 | 87.94 | 4.97 | 0.47 | 0.19 | 0.01 | 0.04 | 4.16 |
| BBB | 0.03 | 0.24 | 4.56 | 84.26 | 4.19 | 0.76 | 0.15 | 0.22 | 5.59 |
| BB | 0.03 | 0.06 | 0.4 | 6.09 | 76.09 | 6.82 | 0.96 | 0.98 | 8.58 |
| B | 0 | 0.09 | 0.29 | 0.41 | 5.11 | 74.62 | 3.43 | 5.3 | 10.76 |
| CCC | 0.13 | 0 | 0.26 | 0.77 | 1.66 | 8.93 | 53.19 | 21.94 | 13.14 |
| D | 0 | 0 | 0 | 0 | 1 | 3.1 | 9.29 | 51.29 | 37.32 |
| N.R. | 0 | 0 | 0 | 0 | 0 | 0.1 | 8.55 | 74.06 | 17.07 |

Transition probabilities are expressed in %.

- Higher ratings are more stable: the diagonal coefficients of the matrix go decreasing.
- Starting from the rating AA it is easier to be downgraded (probability 6.72%) than to be upgraded (probability 0.64%).

# Transition over multiple periods

- The transition matrix is also called a stochastic matrix.
- It defines the probabilities of moving from one value of the state to another in one period.
- The probability of moving from one value of the state to another in two periods is determined by P2 because

$$\begin{aligned}\mathbb{P}[Z_{t+2} = z_j | Z_t = z_i] \\ = \sum_{h=1}^{n} \mathbb{P}[Z_{t+2} = z_j | Z_{t+1} = z_h] \times \mathbb{P}[Z_{t+1} = z_h | Z_t = z_i] \\ = \sum_{h=1}^{n} P_{ih} P_{hj} = P_{ij}^{(2)}\end{aligned}$$

# The unconditional distribution

The probability distribution of $Z_t$ evolves according to $\pi'_{t+1} = \pi'_t P$. Therefore

$$\begin{aligned}
\pi'_1 &= \pi'_0 P \\
\pi'_2 &= \pi'_0 P^2 \\
&\vdots \\
\pi'_k &= \pi'_0 P^k
\end{aligned}$$

The limit for $k \to \infty$ is the *time invariant, stationary, or ergodic* distribution of the Markov chain. It is defined by

$$\pi' = \pi' P \quad \Leftrightarrow \quad (I - P')\pi = 0$$

The limit exist and is independent of the initial distribution $\pi_0$ if $p_{ij}^{(k)} > 0$ for some integer $k \geq 1$.

Example 6:

# Unemployment (cont.)

For the worker who can either be employed or unemployed according to Markov matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \\ 0.05 & 0.95 \end{bmatrix}$$

the stationary distribution $\begin{bmatrix} x & 1-x \end{bmatrix}'$ is the solution to:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix} \right\} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} p & -q \\ -p & q \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then $x = \frac{q}{p+q}$ and the stationary distribution is: $\begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}$. This means that the long run probability of being unemployed is 10%.

# References


Hamilton, James M. (1994). *Time Series Analysis.* Princeton University Press. ISBN: 0-691-04289-6.

Heer, Burkhard and Alfred Maußner (2009). *Dynamic General Equilibrium Modeling. Computational Methods and Applications.* 2nd ed. Springer-Verlag Berlin Heidelberg. 702 pp. ISBN: 978-3-642-03148-9.