

# Lecture 8

Applications of consumer theory

Randall Romero Aguilar, PhD  
Universidad de Costa Rica  
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## 1 The Work-Leisure Decision

### The setup

- There are only two goods: consumption goods  $C$  and time.
- Barter economy: consumer exchanges work time for consumption good.
  - Price of consumption is 1.
  - One hour of work is worth  $w$  units of consumption.
- Consumer is endowed with  $h$  hours, to be used in:
  - leisure:**  $l$  = time used at home
  - work:**  $N^s$  = time exchanged in the market (labor time)
- The time constraint for the consumer is then

$$l + N^s = h$$

which states that leisure time plus time spent working must sum to total time available.

**The consumer's real disposable income**

- For his work, consumer gets  $wN^s = w(h - l)$  units of consumption good.
- Consumer also receives  $\pi$  units of consumption good, in the form of real dividend income.
- Consumer must pay a lump-sum tax amount  $T$  to the government.
- Therefore, the budget constraint is

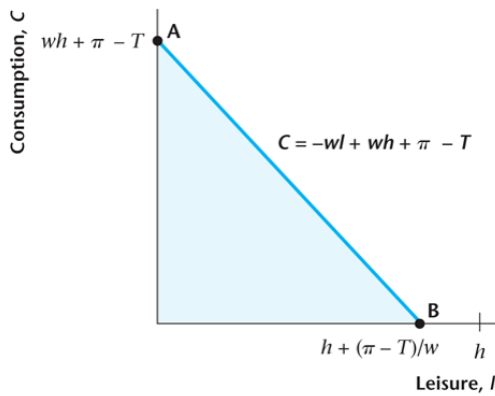
$$C = w(h - l) + \pi - T$$

- which can also be written as

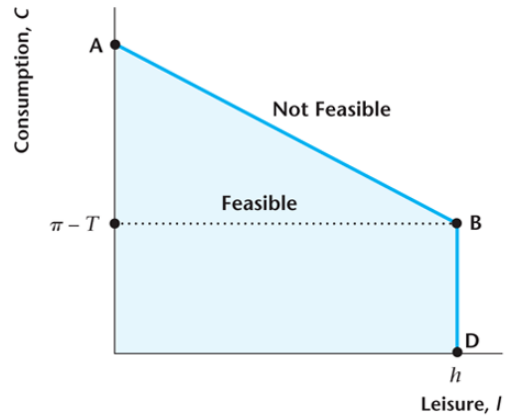
$$C + wl = wh + \pi - T$$

**The budget constraint**

When  $T > \pi$



When  $T < \pi$



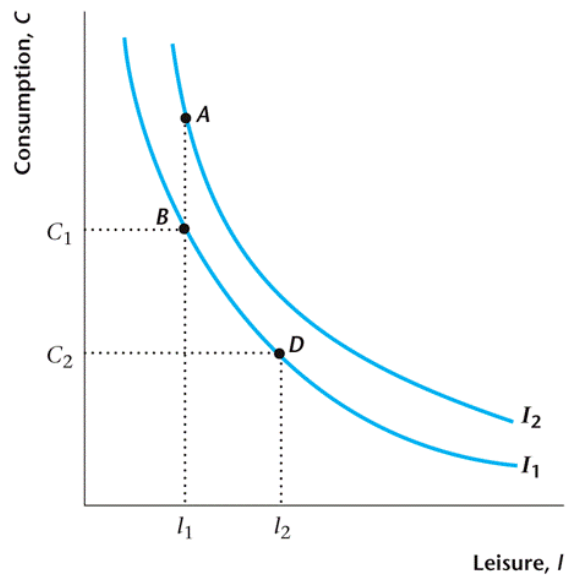
**The consumer's preferences**

The representative consumer's preferences are defined by

$$U(C, l)$$

with  $U(\cdot, \cdot)$  a function that is:

- increasing in both arguments,
- strictly quasiconcave, and
- twice differentiable.



### The consumer's problem

- The consumer's optimization problem is to choose  $C$  and  $l$  so as to maximize  $U(C, l)$  subject to his or her budget constraint—that is,

$$\max_{C, l} U(C, l) \quad \text{s.t.} \quad \begin{cases} C = w(h - l) + \pi - T \\ l \leq h \end{cases}$$

- This problem is a constrained optimization problem, with the associated Lagrangian

$$\mathcal{L} = U(C, l) + \lambda[w(h - l) + \pi - T - C] + \mu(h - l)$$

where  $\lambda$  and  $\mu$  are the Lagrange multipliers.

### Solving the problem

- We assume that there is an interior solution to the consumer's problem where  $C > 0$  and  $0 < l$ .
- This can be guaranteed by assuming that

$$U_C(0, l) = \infty \quad \text{and} \quad U_l(C, 0) = \infty$$

- The first-order conditions are

$$\begin{aligned} U_C(C, l) - \lambda &= 0 \\ U_l(C, l) - \lambda w - \mu &= 0 \\ w(h - l) + \pi - T - C &= 0. \end{aligned}$$

- Slackness conditions:

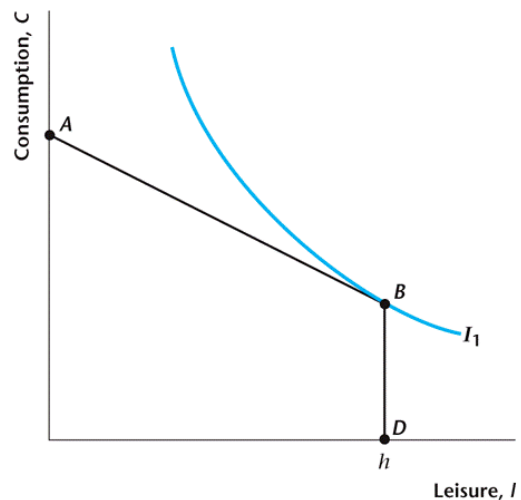
$$\mu \geq 0 \quad h - l \geq 0 \quad \mu(h - l) = 0$$

Case 1:  $l = h$  (consumer does not work!)

- For this case to be feasible, we require that  $C = \pi - T > 0$ .
- From the first two FOCs and nonnegativity of multiplier:

$$\begin{aligned} U_l(\pi - T, h) - wU_C(\pi - T, h) &= \mu \geq 0 \\ \Leftrightarrow w &\leq \frac{U_l(\pi - T, h)}{U_C(\pi - T, h)} \end{aligned}$$

- Thus, consumer does not work if he has  $\pi - T > 0$ , and at bundle  $(\pi - T, h)$  the market wage rate is less than his MRS of leisure for consumption.
- In a competitive equilibrium we cannot have  $l = h$ , as this would imply that nothing would be produced and  $C = 0$ .



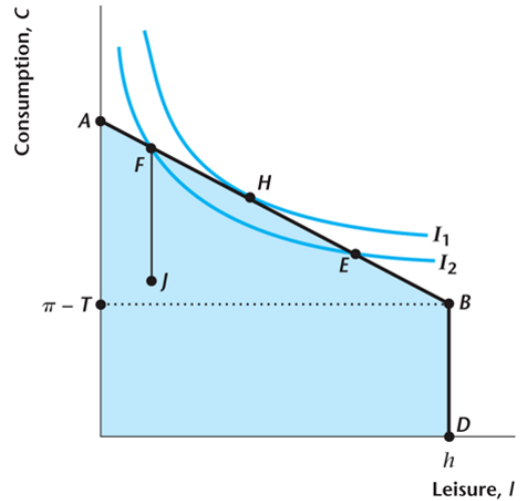
Case 2:  $\mu = 0$  (consumer goes to work!)

- From the first two FOCs:

$$U_l(C^*, l^*) = wU_C(C^*, l^*)$$

$$\Leftrightarrow w = \frac{U_l(C^*, l^*)}{U_C(C^*, l^*)}$$

- Thus, consumer works  $N^{s*} = h - l^*$  hours and consumes  $C^* = w(h - l^*) + \pi - T$ .
- At this allocation, his MRS of leisure for consumption equals the market wage rate.



### A parametric example

$$U(C, l) = \ln(c) + \gamma \ln(l)$$

- FOC

$$MRS_{lC} = \frac{U_l}{U_C} = \frac{\frac{\gamma}{l}}{\frac{1}{C}} = \frac{\gamma C}{l} = w$$

- Time and budget constraints:

$$w = \frac{\gamma C}{h - N^s}$$

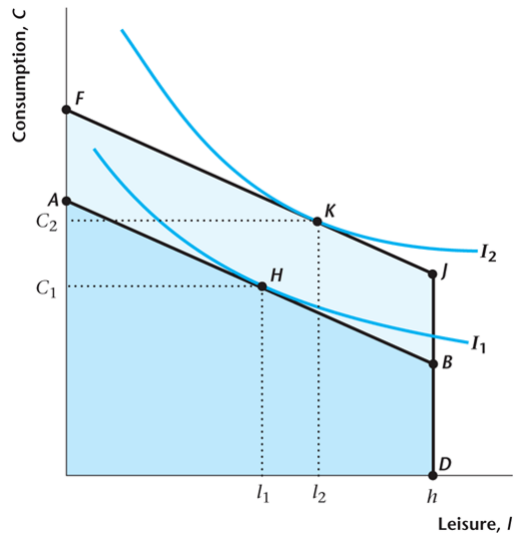
$$C = wN^s + \pi - T$$

- Then

$$N^{s*} = \frac{wh - \gamma(\pi - T)}{(1 + \gamma)w}$$

### Real Dividends or Taxes Change for the Consumer

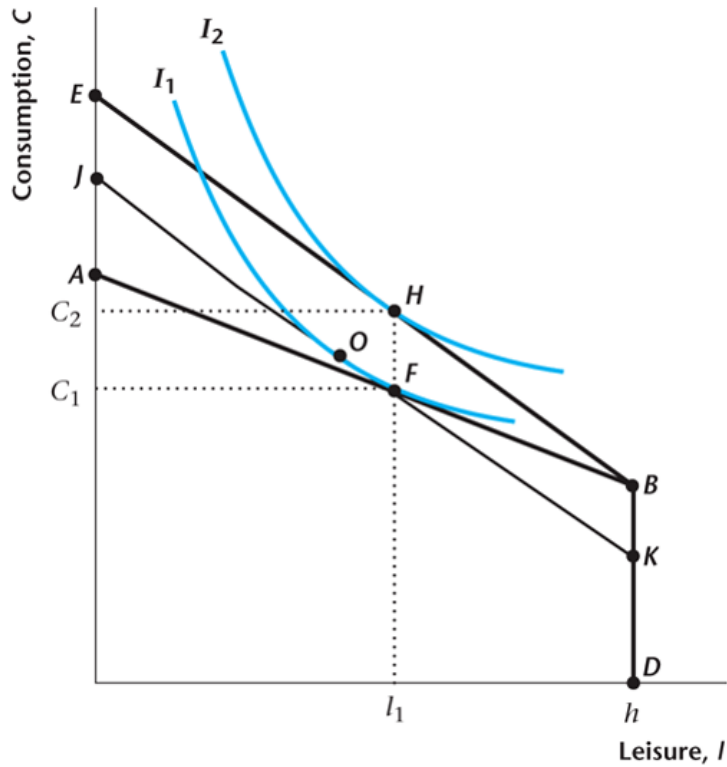
- Assume that consumption and leisure are both normal goods.
- An increase in dividends or a decrease in taxes will then cause the consumer to increase consumption and reduce the quantity of labor supplied (increase leisure).



### An Increase in the Market Real Wage Rate

- This has income and substitution effects.
- *Substitution effect*: the price of leisure rises, so the consumer substitutes from leisure to consumption.
- *Income effect*: the consumer is effectively more wealthy and, since both goods are normal, consumption increases and leisure increases.
- Conclusion: Consumption must rise, but leisure may rise or fall.

### Increase in the Real Wage Rate—Income and Substitution Effects



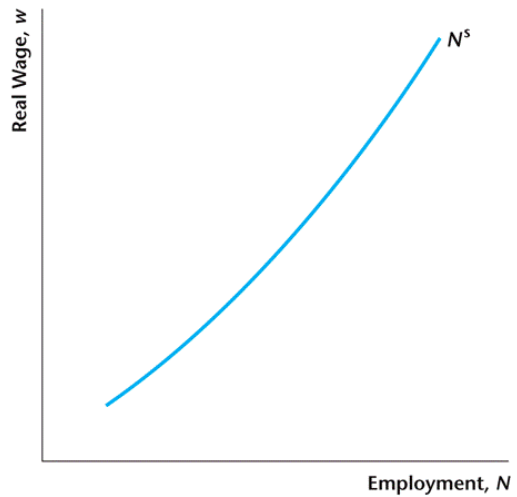
### The labor supply function

- Suppose  $l(w)$  is a function that tells us how much leisure the consumer wishes to consume, given the real wage  $w$ .
- Then, the labor supply curve is given by

$$N^s(w) = h - l(w)$$

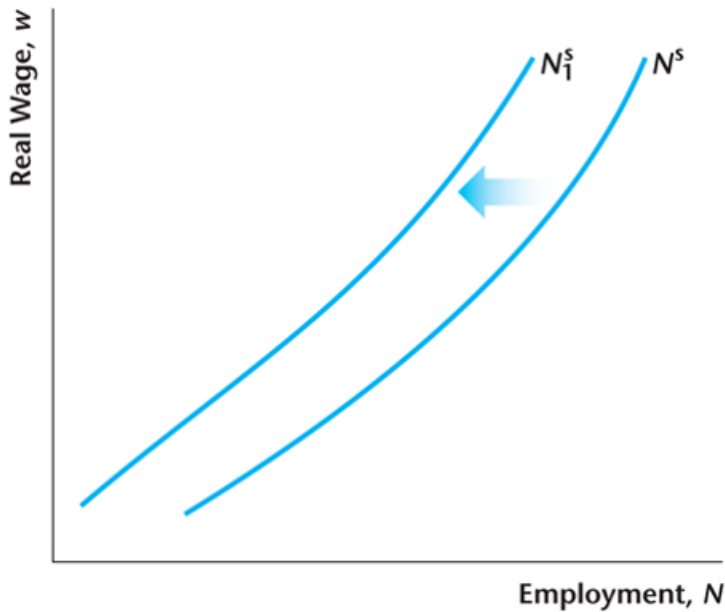
### The slope of the labor supply function

- We do not know whether labor supply is increasing or decreasing in the real wage, because the effect of a wage increase on the consumer's leisure choice is ambiguous.
- Assuming that the substitution effect is larger than the income effect of a change in the real wage, labor supply increases with an increase in the real wage, and the labor supply schedule is upward-sloping.



### Labor supply response to an increase in dividend

- An increase in nonwage disposable income shifts the labor supply curve to the left, that is, from  $N^s$  to  $N_1^s$ , because leisure is a normal good



### 1.1 Comparative statics in leisure-consumption model

#### The economist's problem

- You have a model with  $n$  endogenous variables  $\mathbf{y}$  and  $m$  exogenous variables  $\mathbf{x}$ , whose solution is described by  $\mathbf{y} = \Psi(\mathbf{x})$ .
- You have found  $n$  model conditions of the form  $g(\mathbf{x}, \mathbf{y}) = 0$ .
- *Problem:* How to analyze the comparative statics of the model *without* an explicit formula for  $\Psi(\mathbf{x})$ ?
- *Solution:* compute the total derivative of  $g$ , using the chain rule.

**Side note: The gradient and the Hessian matrix** Let  $f$  be a function,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , where  $\mathbf{x} = (x_1 \ \cdots \ x_n)'$ . We denote the first partial derivatives of  $f(\mathbf{x})$  by

$$f_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i} \quad \text{and} \quad \nabla f(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

and the Hessian matrix of  $f(\mathbf{x})$  by

$$H(\mathbf{x}) = \begin{bmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \cdots & f_{nn}(\mathbf{x}) \end{bmatrix}$$

**Side note: The Jacobian** Let  $f$  be a function,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ :

$$f(\mathbf{x}) = \begin{bmatrix} f^1(\mathbf{x}) \\ \vdots \\ f^m(\mathbf{x}) \end{bmatrix}$$

We denote *the Jacobian* of  $f(\mathbf{x})$  by

$$J(\mathbf{x}) = \begin{bmatrix} f_1^1(\mathbf{x}) & f_2^1(\mathbf{x}) & \dots & f_n^1(\mathbf{x}) \\ f_1^2(\mathbf{x}) & f_2^2(\mathbf{x}) & \dots & f_n^2(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^m(\mathbf{x}) & f_2^m(\mathbf{x}) & \dots & f_n^m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \nabla f^1(\mathbf{x})' \\ \nabla f^2(\mathbf{x})' \\ \vdots \\ \nabla f^m(\mathbf{x})' \end{bmatrix}$$

**Side note: A partitioned Jacobian**

- Let  $g(\mathbf{x}, \mathbf{y})$  be a function of vectors  $\mathbf{x} \in \mathbb{R}^m$  and  $\mathbf{y} \in \mathbb{R}^n$ , such that  $g : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ .
- Think of  $g$  as a system of  $n$  nonlinear equations on  $n$  endogenous variables  $\mathbf{y}$  and  $m$  exogenous variables  $\mathbf{x}$ .
- The partial Jacobians  $Dg_y$  and  $Dg_x$  form a partition of the Jacobian:

$$J(\mathbf{x}, \mathbf{y}) = [Dg_y \mid Dg_x] = \begin{bmatrix} g_{y_1}^1 & g_{y_2}^1 & \dots & g_{y_n}^1 & g_{x_1}^1 & g_{x_2}^1 & \dots & g_{x_m}^1 \\ g_{y_1}^2 & g_{y_2}^2 & \dots & g_{y_n}^2 & g_{x_1}^2 & g_{x_2}^2 & \dots & g_{x_m}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{y_1}^n & g_{y_2}^n & \dots & g_{y_n}^n & g_{x_1}^n & g_{x_2}^n & \dots & g_{x_m}^n \end{bmatrix}$$

**Side note: The total derivative**

- The total derivative of  $g(\mathbf{x}, \mathbf{y})$  satisfies

$$\sum_{i=1}^n \frac{\partial g^k}{\partial y_i} dy_i + \sum_{i=1}^m \frac{\partial g^k}{\partial x_i} dx_i = 0, \quad \forall k = 1, \dots, n$$

- This can be written in terms of the partitioned Jacobian:

$$0 = \begin{bmatrix} g_{y_1}^1 & g_{y_2}^1 & \dots & g_{y_n}^1 \\ g_{y_1}^2 & g_{y_2}^2 & \dots & g_{y_n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ g_{y_1}^n & g_{y_2}^n & \dots & g_{y_n}^n \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{bmatrix} + \begin{bmatrix} g_{x_1}^1 & g_{x_2}^1 & \dots & g_{x_m}^1 \\ g_{x_1}^2 & g_{x_2}^2 & \dots & g_{x_m}^2 \\ \vdots & \vdots & \ddots & \vdots \\ g_{x_1}^n & g_{x_2}^n & \dots & g_{x_m}^n \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix}$$

$$= Dg_y \, d\mathbf{y} + Dg_x \, d\mathbf{x}$$

- Then  $d\mathbf{y} = -[Dg_y]^{-1} Dg_x \, d\mathbf{x}$ , assuming inverse is defined.



## Comparative statics in leisure-consumption model

In our leisure-consumption model, the solution required that:

$$\begin{aligned} g_1(c, l, w, \pi) &= U_l - wU_c = 0 \\ g_2(c, l, w, \pi) &= c - wh + wl - \pi = 0 \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= \begin{bmatrix} g_c^1 & g_l^1 \\ g_c^2 & g_l^2 \end{bmatrix} \begin{bmatrix} dc \\ dl \end{bmatrix} + \begin{bmatrix} g_w^1 & g_\pi^1 \\ g_w^2 & g_\pi^2 \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \\ &= \begin{bmatrix} U_{lc} - wU_{cc} & U_{ll} - wU_{cl} \\ 1 & w \end{bmatrix} \begin{bmatrix} dc \\ dl \end{bmatrix} + \begin{bmatrix} -U_c & 0 \\ l - h & -1 \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \\ \begin{bmatrix} dc \\ dl \end{bmatrix} &= \begin{bmatrix} U_{lc} - wU_{cc} & U_{ll} - wU_{cl} \\ 1 & w \end{bmatrix}^{-1} \begin{bmatrix} U_c & 0 \\ h - l & 1 \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \\ &= \frac{1}{\nabla} \begin{bmatrix} w & wU_{cl} - U_{ll} \\ -1 & U_{lc} - wU_{cc} \end{bmatrix} \begin{bmatrix} U_c & 0 \\ h - l & 1 \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \\ &= \frac{1}{\nabla} \begin{bmatrix} wU_c + (h - l)(wU_{cl} - U_{ll}) & wU_{cl} - U_{ll} \\ -U_c + (h - l)(U_{lc} - wU_{cc}) & U_{lc} - wU_{cc} \end{bmatrix} \begin{bmatrix} dw \\ d\pi \end{bmatrix} \end{aligned}$$

where

$$\nabla = - (w^2 U_{cc} - 2wU_{cl} + U_{ll}) = - \begin{vmatrix} 0 & 1 & w \\ 1 & U_{cc} & U_{cl} \\ w & U_{lc} & U_{ll} \end{vmatrix} \geq 0$$

The comparative statics follows from:

$$\frac{dc}{d\pi} = \frac{wU_{cl} - U_{ll}}{\nabla} > 0 \quad (c \text{ is normal})$$

$$\frac{dc}{dw} = \frac{wU_c + (h - l)(wU_{cl} - U_{ll})}{\nabla} > 0$$

$$\frac{dl}{d\pi} = \frac{U_{lc} - wU_{cc}}{\nabla} > 0 \quad (l \text{ is normal})$$

$$\frac{dl}{dw} = \frac{-U_c + (h - l)(U_{lc} - wU_{cc})}{\nabla} \quad ? \quad 0$$

## 2 Choice under uncertainty

### Choice under uncertainty

- Until now, we have been concerned with the behavior of a consumer under conditions of certainty.
- However, many choices made by consumers take place under conditions of uncertainty.
- In this section we explore how the theory of consumer choice can be used to describe such behavior.

## The choices

- The first question to ask is *what is the basic “thing” that is being chosen?*
- The consumer is presumably concerned with the probability distribution of getting different consumption bundles of goods.
- A probability distribution consists of a list of different outcomes—in this case, consumption bundles—and the probability associated with each outcome.
- When a consumer decides how much automobile insurance to buy or how much to invest in the stock market, he is in effect deciding on a pattern of probability distribution across different amounts of consumption.

## Contingent consumption

- Let us think of the different outcomes of some random event as being different *states of nature*.
- A *contingent consumption plan* is a specification of what will be consumed in each different state of nature.
- Contingent means depending on something not yet certain.
- People have preferences over different plans of consumption, just like they have preferences over actual consumption.
- We can think of preferences as being defined over different consumption plans.

## Utility functions and probabilities

- If the consumer has reasonable preferences about consumption in different circumstances, then we can use a utility function to describe these preferences.
- However, *uncertainty* does add a special structure to the choice problem.
- How a person values consumption in one state as compared to another will depend on the probability that the state in question will actually occur.
- For this reason, we will write the utility function as depending on the probabilities as well as on the consumption levels.

## Utility with discrete random outcomes

- If there are  $n$  possible states of nature  $s$ , then  $c$  is a *discrete random variable* with support  $\{c_1, \dots, c_n\}$ , whose values are realized with probabilities  $\{p_1, \dots, p_n\}$ .

$s$	$\mathbb{P}$	$c$	$u(c)$
1	$\pi_1$	$c_1$	$u(c_1)$
2	$\pi_2$	$c_2$	$u(c_2)$
$\vdots$		$\vdots$	
$n$	$\pi_n$	$c_n$	$u(c_n)$

- Utility is

$$U(c_1, \dots, c_n; \pi_1, \dots, \pi_n) = \sum_{i=1}^n \pi_i u(c_i)$$

## Utility with continuous random outcomes

- If there are infinite states of nature, we think of  $c$  as a *continuous random variable*.
- If  $c$  has support  $\mathbf{C}$ , pdf  $f(c)$  and cdf  $F(c)$ , then utility is

$$\begin{aligned} U(c, f) &= \int_{\mathbf{C}} f(c)u(c) \, dc \\ &= \int_{\mathbf{C}} u(c) \, dF(c) \end{aligned}$$

## von Neumann-Morgenstern utility

- We refer to a utility function  $U$  with the particular form described here as an *expected utility* function, or, sometimes, a *von Neumann-Morgenstern utility* function:

$$U(c, \mathbb{P}) \equiv \mathbb{E} u(c) = \begin{cases} \sum_{i=1}^n \pi_i u(c_i) & \text{discrete} \\ \int_{\mathbf{C}} u(c) \, dF(c) & \text{continuous} \end{cases}$$

- We refer to  $u(c)$  as the *Bernoulli utility* function.

## 2.1 Demand for insurance

### Growing potatoes in uncertain weather

- A farmer grows potatoes for own consumption.
- The weather  $s$  can be *good* or *bad*, affecting the amount of potatoes (real income  $y$ ) he actually harvests:

$s$ (weather)	$\mathbb{P}$	$y$
$g$ (good)	$\pi_g$	$W$
$b$ (bad)	$\pi_b$	$W - L$

- That is, if weather is bad, he loses  $L$  potatoes.
- Expected consumption of potatoes:

$$\mathbb{E} c = \mathbb{E} y = (1 - \pi_b)W + \pi_b(W - L) = W - \pi_b L$$

### An insurance contract

- Farmer can insure  $K$  potatoes, premium is  $\gamma$  per unit.
- Choices are contingent consumption plans:

$s$	$\mathbb{P}$	$y$	insure	$c$
$g$	$\pi_g$	$W$	$-\gamma K$	$W - \gamma K$
$b$	$\pi_b$	$W - L$	$(1 - \gamma)K$	$W - \gamma K + K - L$

### Expected utility of buying insurance coverage $K$

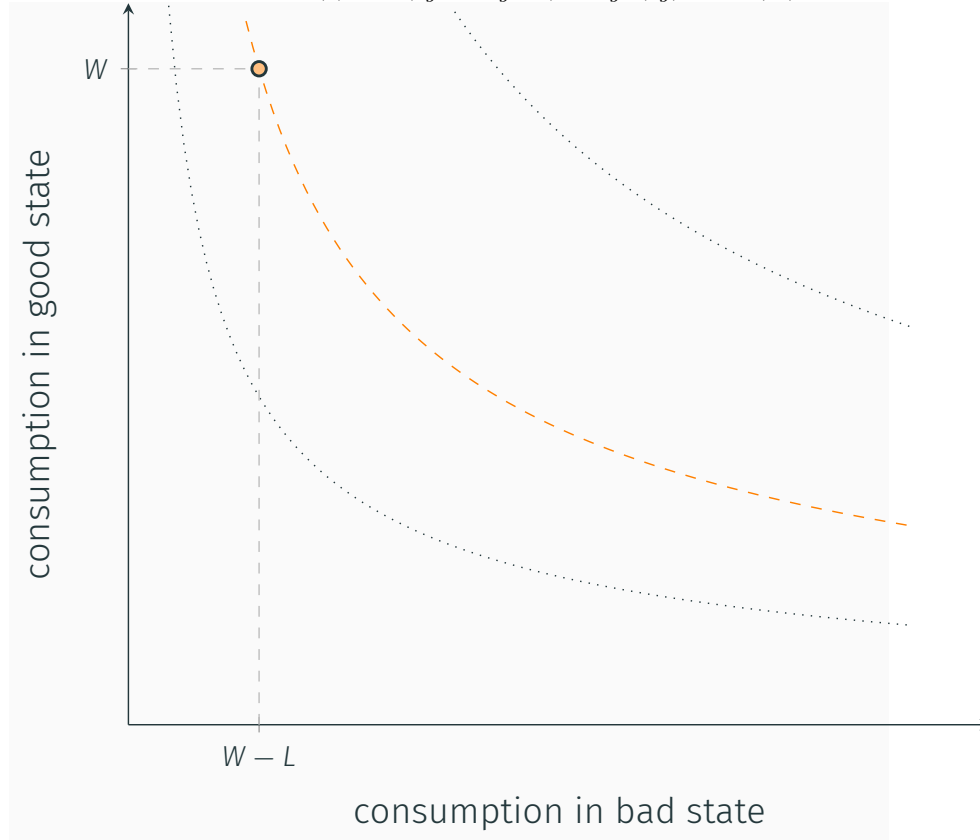
- Expected utility is

$$\begin{aligned} U(c_g, c_b; \pi_g, \pi_b) &\equiv \mathbb{E} u(c) \\ &= \pi_g u(c_g) + \pi_b u(c_b) \\ &= \pi_g u(W - \gamma K) + \pi_b u(W - \gamma K + K - L) \end{aligned}$$

- MRS of bad-weather potatoes for one good-weather potato is

$$MRS_{bg} = \frac{U_{c_b}}{U_{c_g}} = \frac{\pi_b u'(c_b)}{\pi_g u'(c_g)}$$

**Objective function:**  $\mathbb{E} u(c) = U(c_g, c_b, \pi_g, \pi_b) = \pi_g u(c_g) + \pi_b u(c_b)$



**Budget constraint**  $(c_g, c_b) = (y_g - \gamma K, y_b + (1 - \gamma)K)$

- We have

$$K = \frac{y_g - c_g}{\gamma} = \frac{c_b - y_b}{1 - \gamma}$$

- Therefore

$$c_g + \frac{\gamma}{1 - \gamma} c_b = y_g + \frac{\gamma}{1 - \gamma} y_b$$

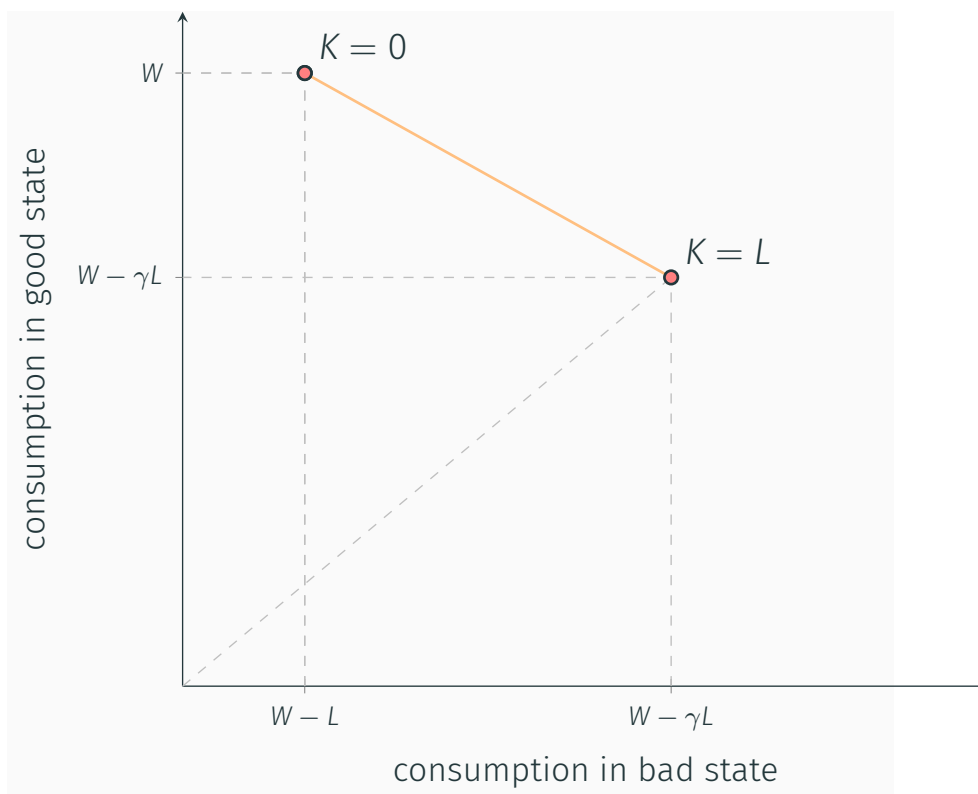
- Substitute  $y_g = W$  and  $y_b = W - L$  to get

$$\begin{aligned} c_g + \frac{\gamma}{1 - \gamma} c_b &= W + \frac{\gamma}{1 - \gamma} (W - L) \\ &= \frac{1}{1 - \gamma} W - \frac{\gamma}{1 - \gamma} L \end{aligned}$$

- The relative price (in terms of potatoes in good weather) of a potato in bad weather is  $p = \frac{\gamma}{1-\gamma}$

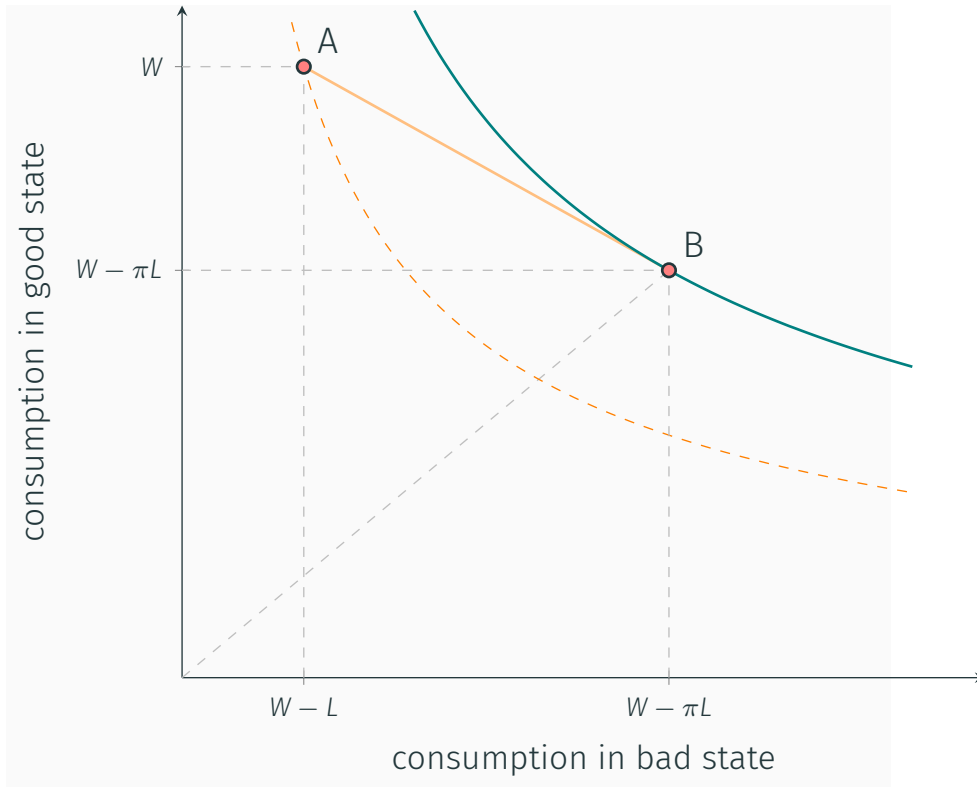
*Budget constraint:*

$$c_g + \frac{\gamma}{1-\gamma}c_b = \frac{1}{1-\gamma}W - \frac{\gamma}{1-\gamma}L$$



*Optimality condition:*

$$MRS_{bg} = \frac{\pi_b u'(c_b)}{\pi_g u'(c_g)} = \frac{\gamma}{1-\gamma} = p$$



### Demand for insurance

- Of course, we could also solve for optimal  $K$  directly:

$$\max_K \{ \pi_g u(W - \gamma K) + \pi_b u(W - L - \gamma K + K) \}$$

- FOC:

$$0 = -\gamma \pi_g u'(W - \gamma K) + (1 - \gamma) \pi_b u'(W - L - \gamma K + K)$$

$$\Leftrightarrow \frac{\pi_b u'(W - L - \gamma K + K)}{\pi_g u'(W - \gamma K)} = \frac{\gamma}{1 - \gamma}$$

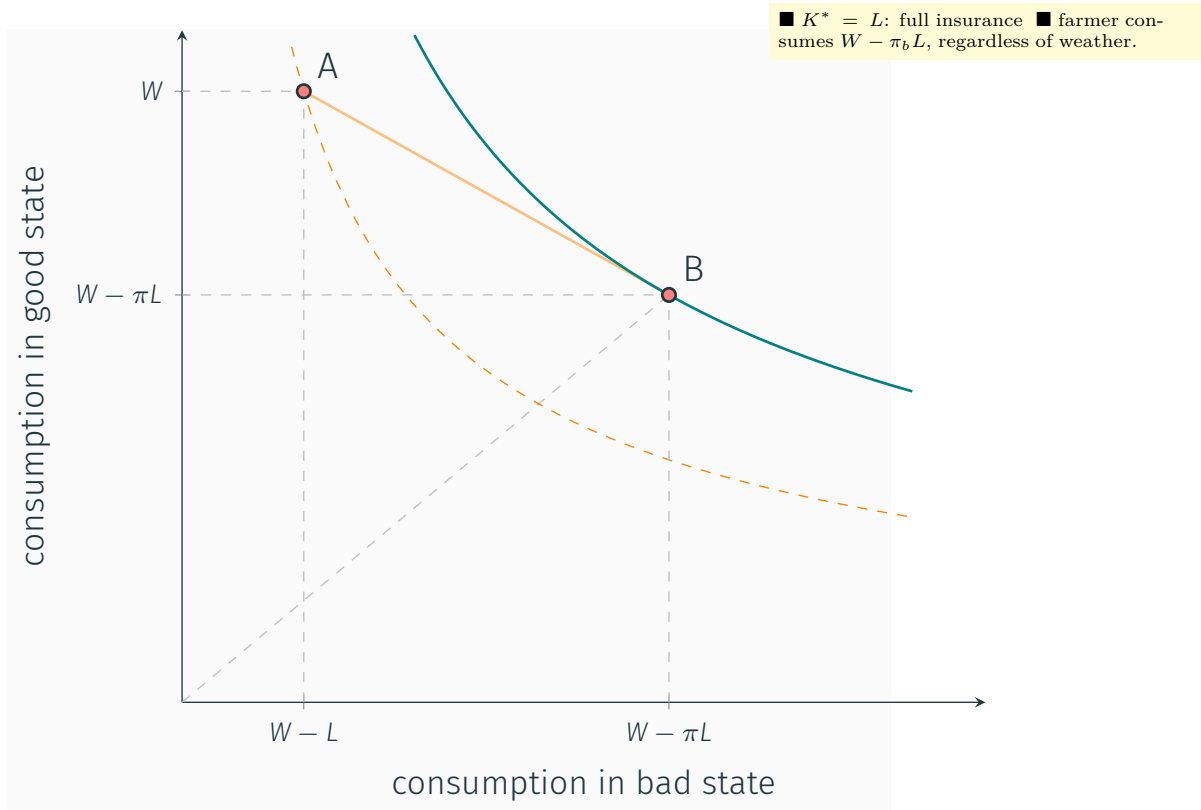
### Risk of losses and price of insurance

- The market price of insurance should satisfy  $\gamma \geq \pi_b$ , so the insurer gets enough revenue  $\gamma K$  to cover expected payments  $\pi_b K$ . This implies that:

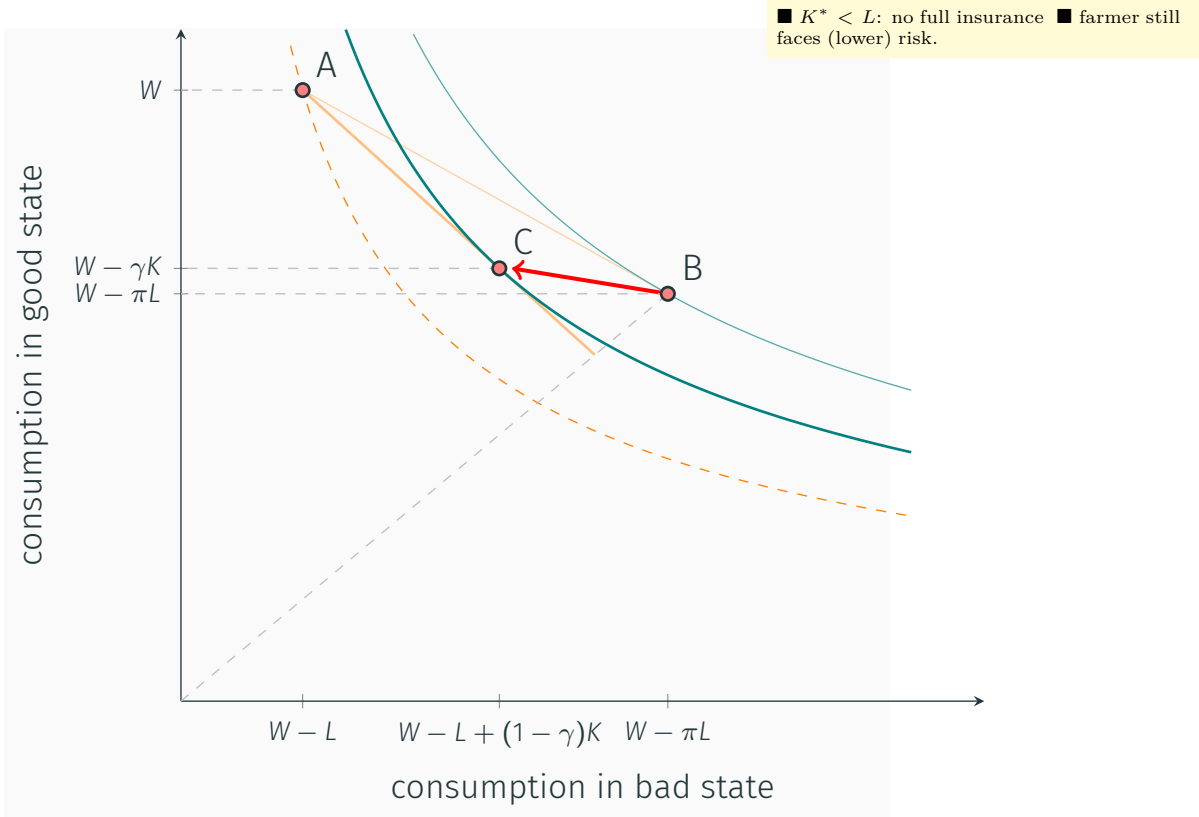
$$\begin{aligned} \gamma &\geq \pi_b \\ 1 - \pi_b &\geq 1 - \gamma \\ \gamma(1 - \pi_b) &\geq \pi_b(1 - \gamma) \\ 1 &\geq \frac{\pi_b(1 - \gamma)}{\gamma(1 - \pi_b)} = \frac{u'(c_g)}{u'(c_b)} && \text{(from FOC)} \\ u'(c_b) &\geq u'(c_g) \\ c_b &\leq c_g && \text{(assuming risk aversion)} \end{aligned}$$

- Consumer gets full insurance iff it's actuarially fair.

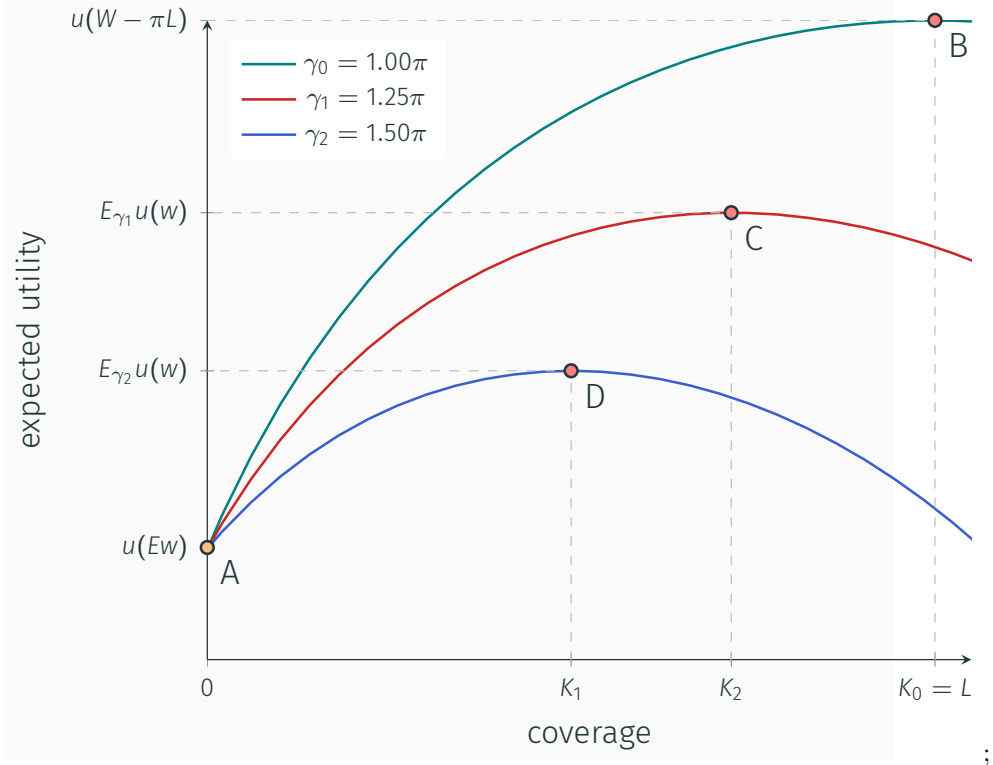
Case  $\gamma = \pi_b$ : actuarially fair insurance



Case  $\gamma > \pi_b$ : insurer expects a profit



Increasing the premium: As insurance gets expensive, consumer buys less coverage.





## Example 1: Logarithmic utility

Let's now assume that  $u(c) = \ln(c)$  From the FOC:

$$\gamma\pi_g u'(W - \gamma K) = (1 - \gamma)\pi_b u'(W - L + (1 - \gamma)K)$$

$$\gamma\pi_g [W - L + (1 - \gamma)K] = (1 - \gamma)\pi_b (W - \gamma K)$$

$$\pi_g \gamma (W - L) + \pi_g \gamma (1 - \gamma) K = \pi_b (1 - \gamma) W - \pi_b \gamma (1 - \gamma) K$$

$$(\pi_b + \pi_g)(1 - \gamma)\gamma K = (\pi_b - \gamma(\pi_b + \pi_g))W + \gamma\pi_g L$$

$$\gamma(1 - \gamma)K = (\pi_b - \gamma)W + \gamma(1 - \pi_b)L$$

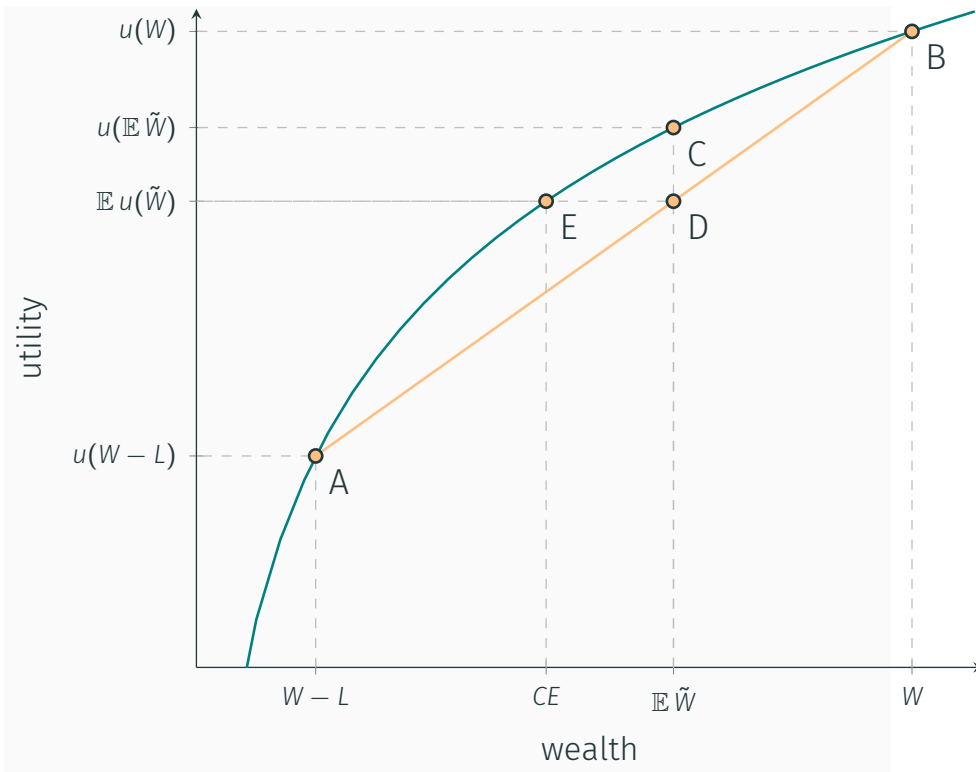
$$K^* = \frac{1 - \pi_b}{1 - \gamma} L - \frac{\gamma - \pi_b}{\gamma(1 - \gamma)} W$$

Optimal contingent consumption plans:

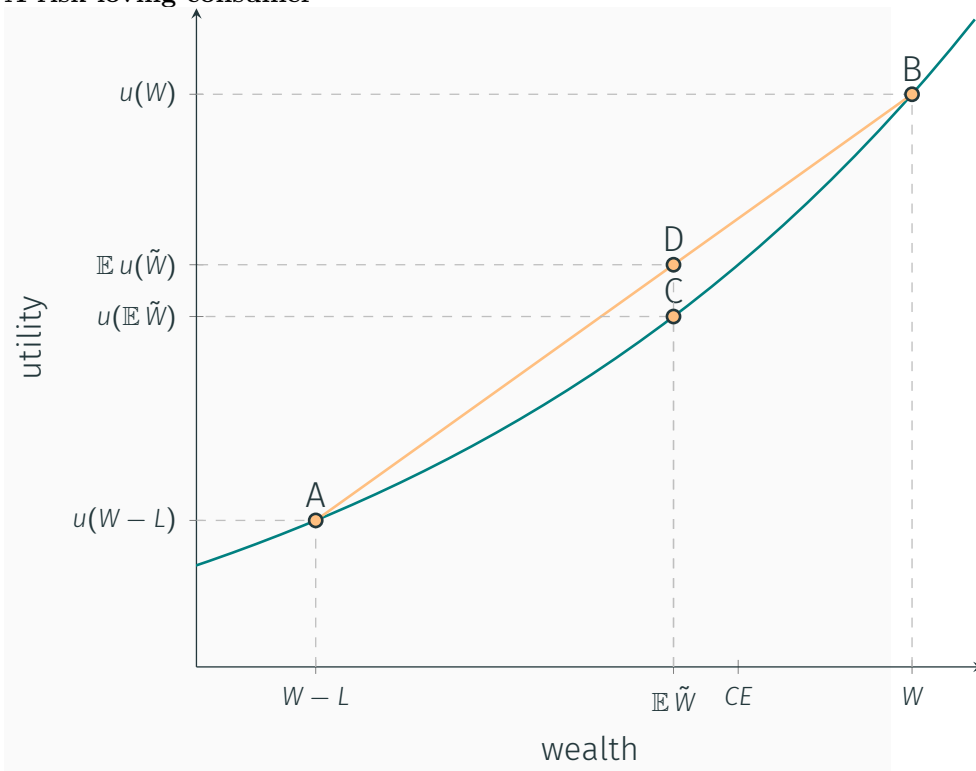
$s$	$\mathbb{P}$	$y$	$c^*$
$g$	$\pi_g$	$W$	$\frac{1 - \pi_b}{1 - \gamma} (W - \gamma L)$
$b$	$\pi_b$	$W - L$	$\frac{\pi_b}{\gamma} (W - \gamma L)$

## 2.2 Risk aversion

A risk averse consumer



**A risk loving consumer**



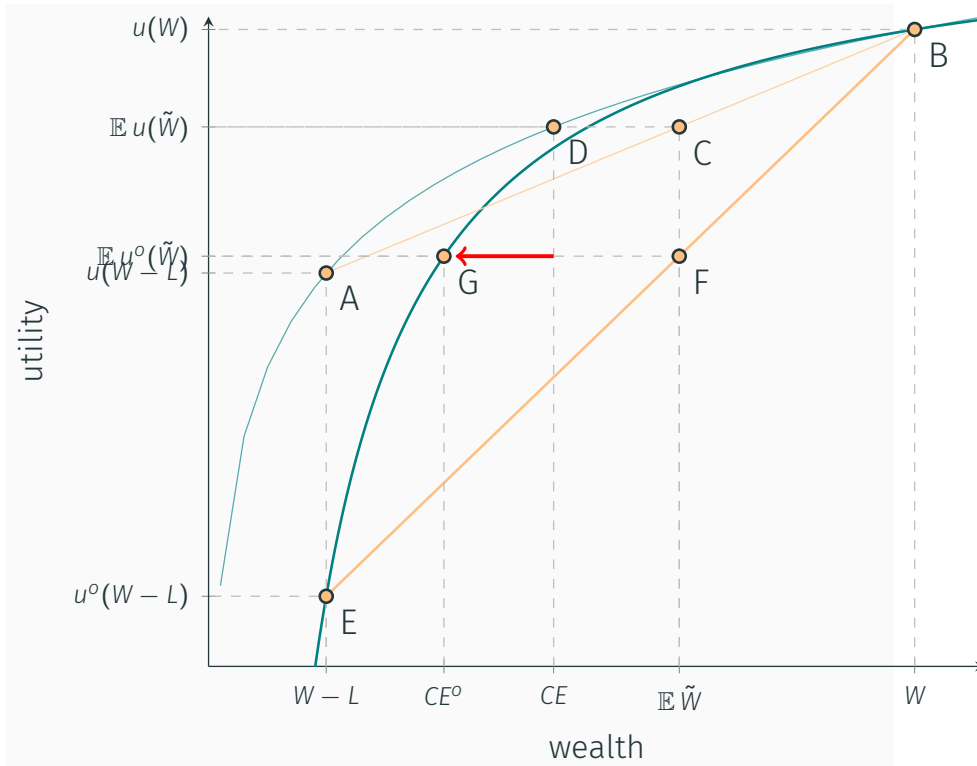
**Measuring risk aversion**

- A consumer with a von Neumann-Morgenstern utility function can be one of the following:
  - *Risk-averse*, with a concave utility function;
  - *Risk-neutral*, with a linear utility function, or;
  - *Risk-loving*, with a convex utility function.
- Then, the degree of risk-aversion a consumer displays would be related to the curvature of their Bernoulli utility function  $u(W)$ .
- The more “curved” a concave  $u(W)$  is, the lower will be a consumer’s certainty equivalent, and the higher their risk premium.
- How do we measure the curvature of a function?
- Simple - *using the function’s second derivative*.

**Arrow-Pratt measure of risk aversion**

Absolute	Relative
$\frac{-u''(W)}{u'(W)}$	$\frac{-u''(W)W}{u'(W)}$
CARA	CRRA
$u(c) = -e^{-\rho c}$	$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$

**A change in risk aversion**



### 2.3 A risky asset

#### A risky asset

- Consider a simple two-period portfolio problem involving two assets, one with a risky (gross) return  $\tilde{R} \geq 0$  and one with a sure (gross) return  $R_f \geq 1$ .
- Let  $w$  be initial wealth, and let  $x \in [0, 1]$  be the share of wealth invested in the risky asset.

$s$	$\mathbb{P}$	risky	risk-free	$c$
$\tilde{R} = R$	$f(R)$	$xw$	$(1-x)w$	$[(1-x)R_f + x\tilde{R}]w$

- In this case the second-period wealth can be written as

$$\begin{aligned}\tilde{W} &= (1-x)R_f w + x\tilde{R}w \\ &= [(1-x)R_f + x\tilde{R}]w\end{aligned}$$

- Note that  $\tilde{W}$  is a random variable since  $\tilde{R}$  is random.

#### Expected utility

- The expected utility from investing  $x$  in the risky asset:

$$v(x) = \mathbb{E} u(c) = \mathbb{E} u\left([(1-x)R_f + x\tilde{R}]w\right)$$

- The portfolio problem is then to choose  $x \in [0, 1]$  to maximize  $v(x)$ :

$$\mathcal{L}(x, \mu, \lambda) = \mathbb{E} u \left( [(1-x)R_f + x\tilde{R}]w \right) + \mu x + \lambda(1-x)$$

- Conditions:

$$\mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f)w \right\} + \mu - \lambda = 0$$

$$\begin{array}{lll} \mu \geq 0 & x \geq 0 & \mu x = 0 \\ \lambda \geq 0 & x \leq 1 & \lambda(1-x) = 0 \end{array}$$

### Second order condition

- Notice that second derivative is

$$\mathbb{E} \left\{ u''(\tilde{W})(\tilde{R} - R_f)^2 w^2 \right\} < 0 \quad \text{iif} \quad u''(\tilde{W}) < 0$$

- SOC requires that consumer is risk-averse.

### Slackness conditions

- The slackness conditions (SC) imply:

- if  $x = 0$ ,  $2^{nd}$  group of SC satisfied with  $\lambda = 0$ .
- if  $x = 1$ ,  $1^{st}$  group of SC satisfied with  $\mu = 0$ .
- if  $0 < x < 1$ , both groups of SC satisfied with  $\lambda = \mu = 0$ .

- Then, we only need to analyze 3 cases:

- $x = 0 \Rightarrow \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} = -\mu \leq 0$
- $x = 1 \Rightarrow \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} = \lambda \geq 0$
- $0 < x < 1 \Rightarrow \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} = 0$

Case 1:  $x = 0 \Rightarrow \tilde{W} = wR_f$

$$\begin{aligned} \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} &\leq 0 \\ \mathbb{E} \left\{ u'(\tilde{W})\tilde{R} \right\} &\leq \mathbb{E} \left\{ u'(\tilde{W})R_f \right\} \\ \mathbb{E} \left\{ u'(wR_f)\tilde{R} \right\} &\leq \mathbb{E} \left\{ u'(wR_f)R_f \right\} \\ u'(wR_f) \mathbb{E} \left\{ \tilde{R} \right\} &\leq u'(wR_f)R_f \\ \mathbb{E} \left\{ \tilde{R} \right\} &\leq R_f \end{aligned}$$

Consumer does not invest in risky asset if its expected return is lower than the risk-free return.

Case 2:  $x = 1 \Rightarrow \tilde{W} = w\tilde{R}$

$$\begin{aligned}
\mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} &\geq 0 \\
\mathbb{E} \left\{ u'(\tilde{W})\tilde{R} \right\} &\geq \mathbb{E} \left\{ u'(\tilde{W})R_f \right\} \\
\mathbb{E} \left\{ u'(w\tilde{R})\tilde{R} \right\} &\geq \mathbb{E} \left\{ u'(w\tilde{R})R_f \right\} \\
R_f &\leq \frac{\mathbb{E} \left\{ u'(w\tilde{R})\tilde{R} \right\}}{\mathbb{E} \left\{ u'(w\tilde{R}) \right\}}
\end{aligned}$$

Consumer does not invest in risk-free asset if its return is “too low”. We need more details about the  $\tilde{R}$  process and utility  $u$  to determine what “too low” is.

Case 3:  $0 < x < 1$

$$\begin{aligned}
0 &= \mathbb{E} \left\{ u'(\tilde{W})(\tilde{R} - R_f) \right\} \\
&= \text{Cov} \left[ u'(\tilde{W}), \tilde{R} - R_f \right] + \mathbb{E} \left[ u'(\tilde{W}) \right] \mathbb{E} \left[ \tilde{R} - R_f \right]
\end{aligned}$$

Then

$$\mathbb{E} \tilde{R} - R_f = \frac{-\text{Cov} \left[ u'(\tilde{W}), \tilde{R} \right]}{\mathbb{E} u'(\tilde{W})} > 0$$

### Example 2: “Investing” in “Tiempos”

- In “Tiempos” lottery, you pick one number out of 100, all of them with equal probability (1%) of winning.
- In *winning* state, your gross return is  $\tilde{R} = 72$ .
- In *losing* state, your gross return is  $\tilde{R} = 0$ .
- If you don’t play, you keep your money ( $R_f = 1$ ).
- Expected return on lottery is

$$\mathbb{E} \tilde{R} = 0.99 \times 0 + 0.01 \times 72 = 0.7128 < 1 = R_f$$

- Therefore, a risk-averse consumer would never play “Tiempos”.

## 3 Intertemporal consumption

### Adding a time dimension

- So far we have only studied static choices.
- Life is full of intertemporal choices: Should I study for my test today or tomorrow? Should I save or should I consume now?
- We will present a simple model: the Life-Cycle/Permanent Income Model of Consumption.

- Developed by Modigliani (Nobel winner 1985) and Friedman (Nobel winner 1976).
- Will allow us to address several key issues: effects of government programs including Social Security, government debts and deficits.

### The model

- Representative household lives 2 periods.
- Utility function:

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

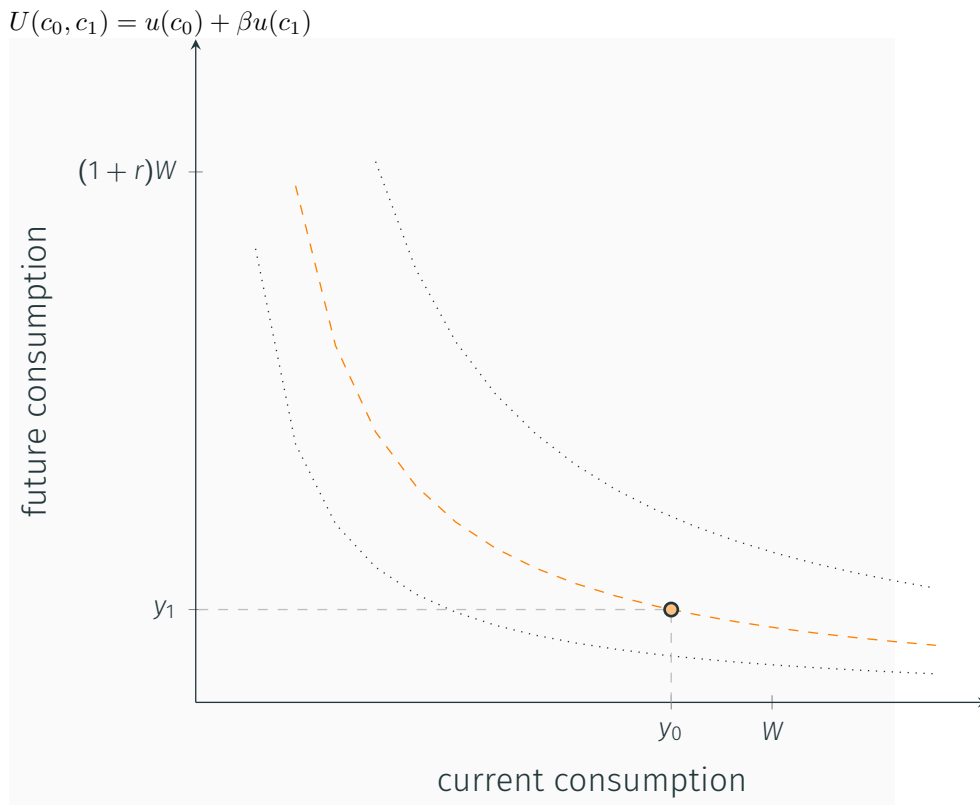
- $c_0$  is consumption in first (current) period of life,
- $c_1$  is consumption in second (future) period of life,
- $0 < \beta < 1$  measures household's degree of impatience.
- Preferences over  $c_0, c_1$  satisfy monotonicity ( $u' > 0$ ) and convexity ( $u'' < 0$ ).

### More on preferences

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

- Consumption smoothing motive, partially offset by discounting.
- Assume  $c_0$  and  $c_1$  are normal: more income  $\Rightarrow$  more of both.
- Intertemporal marginal rate of substitution measures willingness to substitute consumption over time:

$$MRS_{c_0, c_1} = \frac{U_{c_0}(c_0, c_1)}{U_{c_1}(c_0, c_1)} = \frac{u'(c_0)}{\beta u'(c_1)}$$



### Budget constraint I

- Abstract from labor/leisure tradeoff.
- (Labor) income  $y_t \geq 0$  in period  $t = 0, 1$ .
- Initial wealth  $a_0 \geq 0$ .
- Consumer can save part of income or initial wealth in the first period, or it can borrow against future income  $y_1$ .
- Interest rate on both savings and on loans is equal to  $r$ . Gross interest rate  $R \equiv 1 + r$
- Let  $s_t = y_t - c_t$  denote saving.
- Budget constraint in first period:

$$a_1 = R(a_0 + s_0)$$

- Budget constraint in second period:

$$a_2 = R(a_1 + s_1) = 0$$

### Budget constraint (II)

- Combining both constraints:

$$R(a_0 + s_0) + s_1 = 0 \quad \Rightarrow \quad -s_0 - \frac{s_1}{R} = a_0$$

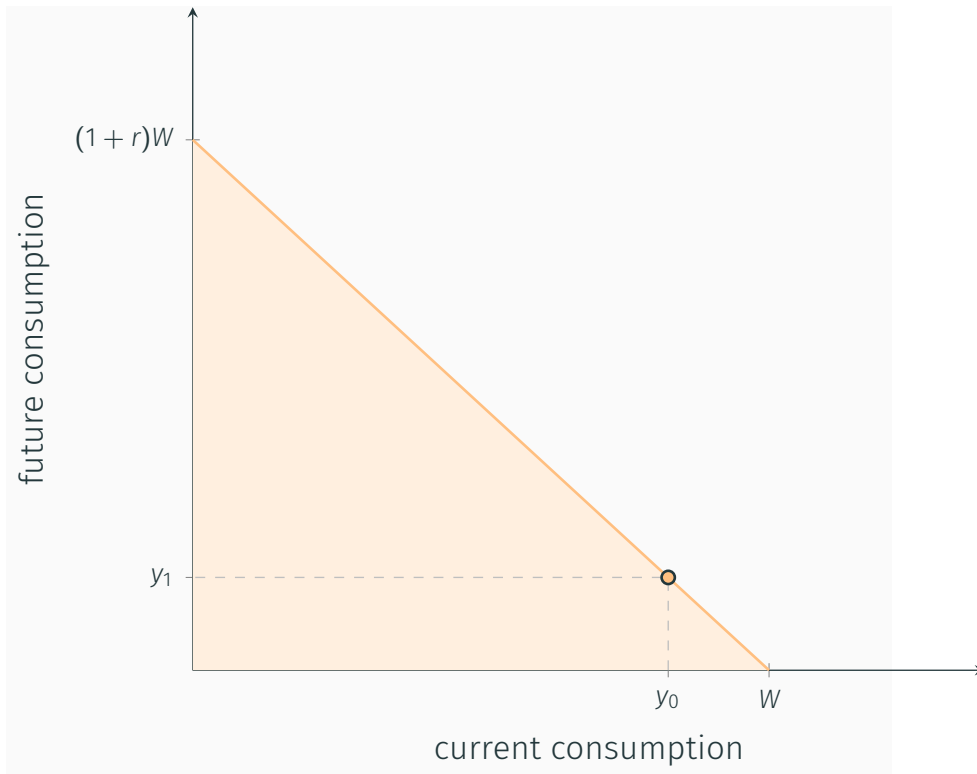
- Substitute  $s_t = y_t - c_t$

$$c_0 + \frac{c_1}{R} = y_0 + \frac{y_1}{R} + a_0 = H + a_0 \equiv W \quad (\text{PVBC})$$

- We have normalized the price of the consumption good in the first period to 1.
- Gross interest rate  $R \equiv 1 + r$  is the relative price of consumption goods today to consumption goods tomorrow.
- Called the *present value budget constraint* (PVBC).

$$c_0 + \frac{c_1}{R} = W$$





**The consumer's problem**

$$\max_{c_0, c_1} \{u(c_0) + \beta u(c_1)\} \quad \text{subject to } c_0 + \frac{c_1}{R} = W$$

- Form Lagrangian with multiplier  $\lambda \geq 0$

$$\mathcal{L}(c_0, c_1, \lambda) = u(c_0) + \beta u(c_1) + \lambda \left( W - c_0 - \frac{c_1}{R} \right)$$

- FOCs:

$$\begin{aligned} u'(c_0) &= \lambda \\ \beta u'(c_1) &= \frac{\lambda}{R} \end{aligned}$$

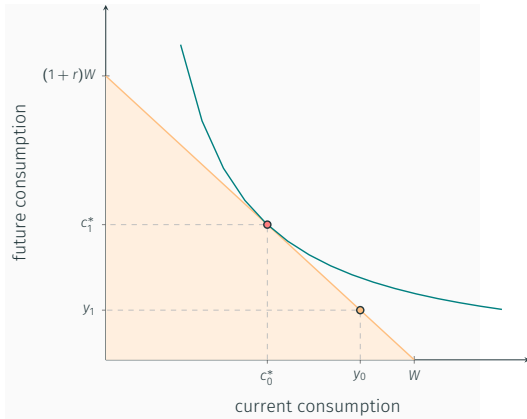
- Combine to get

Euler equation

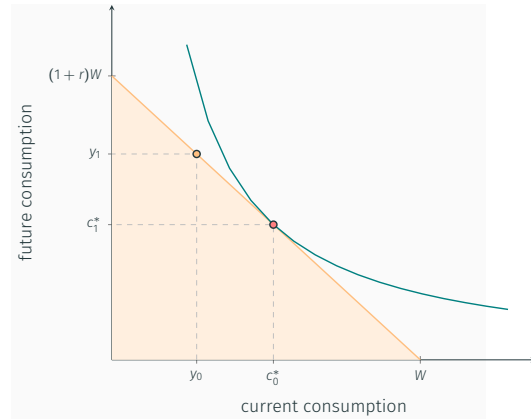
$$u'(c_0) = \beta R u'(c_1)$$

$$u'(c_0) = \beta R u'(c_1)$$

Consumer is a lender



Consumer is a borrower



### Implications of the Euler equation

$$u'(c_0) = \beta R u'(c_1)$$

- Can also be written

$$MRS_{c_0, c_1} = 1 + r$$

- Recall that  $u$  is concave, so  $u'' < 0 \Rightarrow u'(c)$  is decreasing. So if:

$$- \beta(1+r) > 1 \Rightarrow u'(c_0) > u'(c_1) \Rightarrow c_0 < c_1$$

$$- \beta(1+r) < 1 \Rightarrow u'(c_0) < u'(c_1) \Rightarrow c_0 > c_1$$

$$- \beta(1+r) = 1 \Rightarrow u'(c_0) = u'(c_1) \Rightarrow c_0 = c_1$$

- Behavior of consumption over time depends on rate of time preference relative to interest rate.
- If equal, perfect consumption smoothing.

### Example 3: Logarithmic utility

$$u(c) = \ln(c)$$

- Euler equation:

$$\frac{1}{c_0} = \frac{\beta R}{c_1} \Rightarrow c_1 = \beta R c_0$$

- Using the PVBC

$$c_0 = W - \frac{c_1}{R} = W - \beta c_0$$

- So that

$$c_0 = \frac{1}{1+\beta} W$$

$$s_0 = \frac{1}{1+\beta} \left( \beta y_0 - a_0 - \frac{y_1}{R} \right)$$

$$c_1 = \frac{\beta R}{1+\beta} W$$

$$a_1 = \frac{1}{1+\beta} [\beta R(y_0 + a_0) - y_1]$$

- Value function:

$$V(W, r) = (1 + \beta) \ln W + \beta \ln R + \beta \ln \beta - (1 + \beta) \ln(1 + \beta)$$

- Increasing wealth  $W$ , regardless of source, increases consumer utility:

$$\frac{\partial V}{\partial W} = \frac{1 + \beta}{W}$$

- Effect of a change in interest rate  $r$  depends on wealth composition, which in turn determines whether the consumer has positive or negative assets  $a_1$  at the end of period 1:

$$\begin{aligned} \frac{\partial V}{\partial r} &= \frac{1}{R^2 W} [\beta R(y_0 + a_0) - y_1] \\ &= \frac{1 + \beta}{R^2 W} a_1 \end{aligned}$$

### Example 4: CRRA utility

- The logarithmic utility from last example is just a special case of the constant relative risk aversion (CRRA) utility, when  $\sigma = 1$ .

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

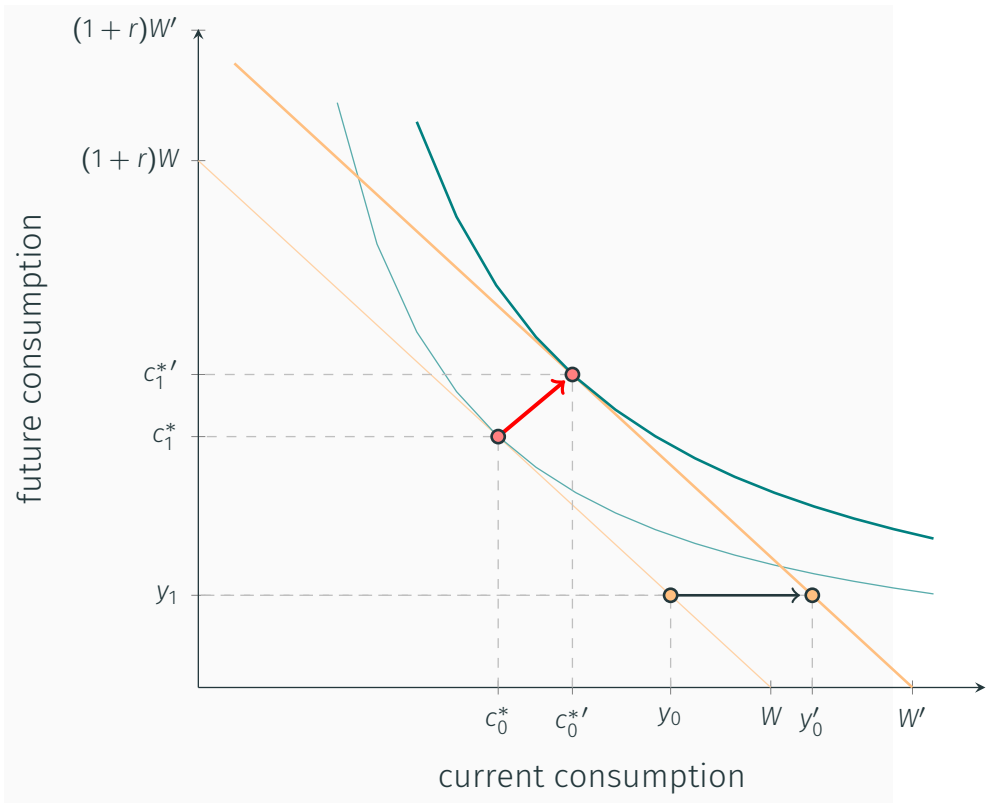
- With CRRA utility, the Bellman equation becomes

$$c_0^{-\sigma} = \beta R c_1^{-\sigma} \quad \Rightarrow \quad c_1 = (\beta R)^{1/\sigma} c_0$$

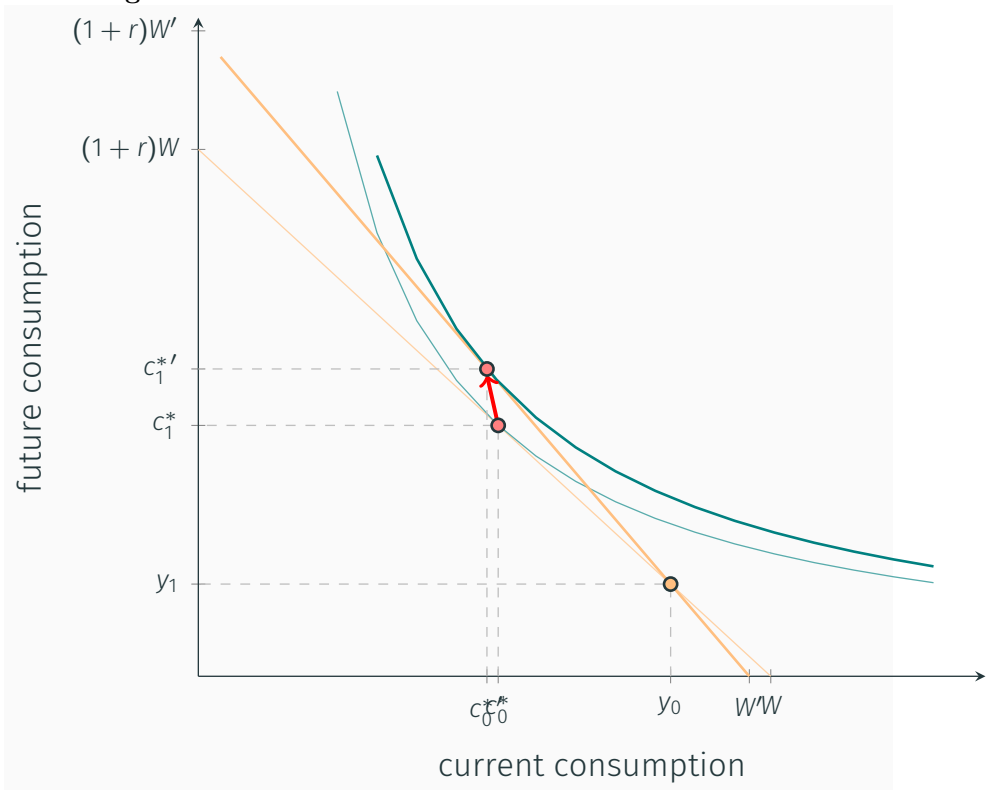
- Use budget constraint  $c_0 + \frac{c_1}{R} = W$  to solve for  $c_0$  and  $c_1$ :

$$c_0 = \frac{R}{R + (\beta R)^{1/\sigma}} W \quad c_1 = \frac{R(\beta R)^{1/\sigma}}{R + (\beta R)^{1/\sigma}} W$$

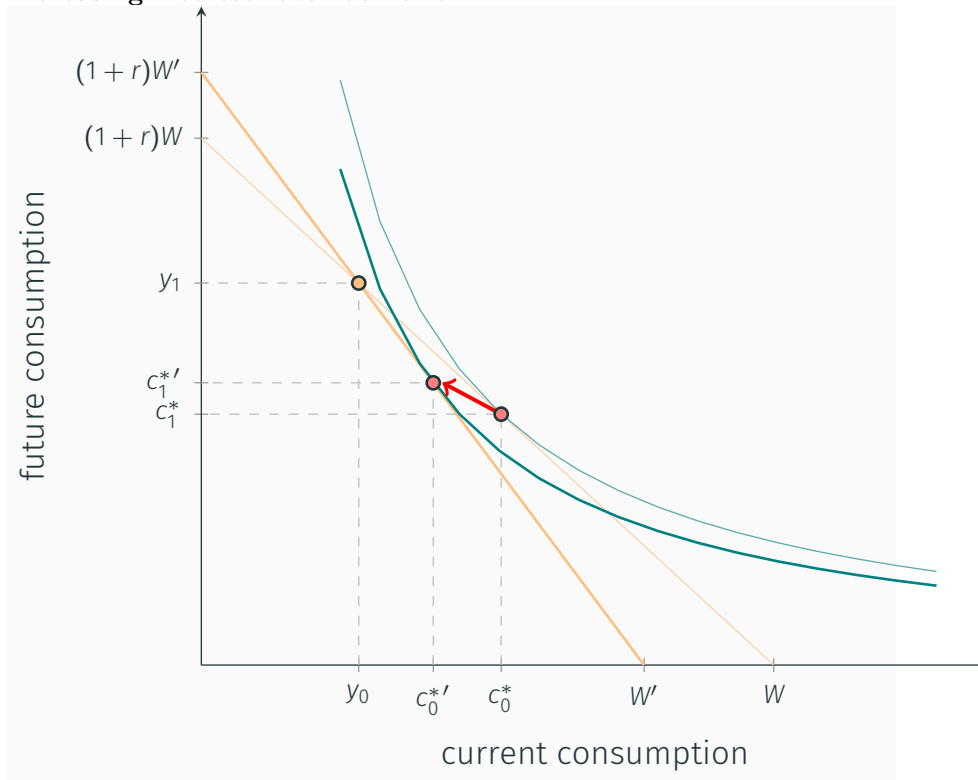
**Increasing wealth**



**Increasing interest rate: lender**



### Increasing interest rate: borrower



## 3.1 Many goods, two time periods

### The model

- A consumer lives two periods, and chooses among  $n + 1$  goods in each period:  $x_{it}$  for  $i \in \{0, 1, \dots, n\}$  and  $t \in \{0, 1\}$ .
- Utility function depends on  $2n + 2$  goods:

$$U = \frac{(\alpha_0 x_{00}^\rho + \dots + \alpha_n x_{n0}^\rho)^{\frac{1-\gamma}{\rho}}}{1-\gamma} + \beta \frac{(\alpha_0 x_{01}^\rho + \dots + \alpha_n x_{n1}^\rho)^{\frac{1-\gamma}{\rho}}}{1-\gamma}$$

- Let  $\mathbf{x}_t$  be the bundle of goods consumed at time  $t$ :

$$\mathbf{x}_t = [x_{0t}, x_{1t}, \dots, x_{nt}]$$

### Constraints in nominal terms

- Consumer can save and borrow money at nominal interest rate  $i$ .
- The budget constraint says that the present value of all consumption purchases must equal the present value of nominal income  $Y_t$ :

$$\sum_{k=0}^n p_{k0} x_{k0} + \frac{1}{1+i} \sum_{k=0}^n p_{k1} x_{k1} = Y_0 + \frac{Y_1}{1+i}$$

- Let  $C_t = \sum_{k=0}^n p_{kt} x_{kt}$  be nominal consumption at time  $t$ .

- Budget constraint becomes

$$C_0 + \frac{C_1}{1+i} = Y_0 + \frac{Y_1}{1+i} \equiv W$$

where  $W$  is nominal wealth.

### Constraints in real terms

- Let  $P_t = (\alpha_0^\sigma p_{0t}^{1-\sigma} + \dots + \alpha_n^\sigma p_{nt}^{1-\sigma})^{\frac{1}{1-\sigma}}$  be the price index at time  $t$
- Notice that  $\frac{P_1}{P_0(1+i)} = \frac{1+\pi}{1+i} = \frac{1}{1+r}$ , where  $\pi$  is the inflation rate, and  $r$  the real interest rate.
- Divide budget constraint by price index  $P_0$

$$\begin{aligned} \frac{C_0}{P_0} + \frac{P_1}{P_0(1+i)} \frac{C_1}{P_1} &= \frac{Y_0}{P_0} + \frac{P_1}{P_0(1+i)} \frac{Y_1}{P_1} = \frac{W}{P_0} \\ c_0 + \frac{c_1}{1+r} &= y_0 + \frac{y_1}{1+r} = w \end{aligned}$$

where  $c_t$  is real consumption,  $y_t$  is real income, and  $w$  is real wealth.

- Constraint says that present value of real (composite) consumption equals the present value of real income.

### Solving the problem: 2 steps

- Let  $\tilde{U}$  denote CES function:  $\tilde{U}(\mathbf{x}_t) = (\alpha_0 x_{0t}^\rho + \dots + \alpha_n x_{nt}^\rho)^{\frac{1}{\rho}}$
- Utility becomes:

$$U = \frac{\tilde{U}(\mathbf{x}_0)^{1-\gamma}}{1-\gamma} + \beta \frac{\tilde{U}(\mathbf{x}_1)^{1-\gamma}}{1-\gamma}$$

- Consumer has to choose  $2n + 2$  variables, subject to 1 budget constraint.
- To solve this problem, consumer makes decisions in two stages
  - *Intra-temporal stage*: Given all prices and the total amount to spend in each period, consumer chooses goods for each period separately.
  - *Inter-temporal stage*: Taking the intra-temporal solution as given, solve the inter-temporal problem:

### Intra-temporal stage

- Given all prices and the total amount to spend in each period, consumer chooses goods for each period separately.
- Since intra-temporal preferences are CES, we know (from Example 4 in Lecture 7) that if consumer spends  $C_t$  dollars and price level is  $P_t$ , the optimal utility he can get is

$$\tilde{V}(C_t, P_t) \equiv \max_{\mathbf{x}_t} \tilde{U}(\mathbf{x}_t) = \frac{C_t}{P_t} = c_t$$

### Inter-temporal stage

- Taking the intra-temporal solution as given, problem becomes:

$$\max_{c_0, c_1} \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \frac{c_1^{1-\gamma}}{1-\gamma} \quad \text{s.t.} \quad c_0 + \frac{c_1}{R} = w$$

- But this is equivalent to what we solved in Example 4 in this lecture. Its solution is characterized by the *Euler equation*

$$c_0^{-\gamma} = \beta R c_1^{-\gamma} \quad \Rightarrow \quad c_1 = (\beta R)^{1/\gamma} c_0$$

- Solution is

$$c_0 = \frac{R}{R + (\beta R)^{1/\gamma}} w \quad c_1 = \frac{R(\beta R)^{1/\gamma}}{R + (\beta R)^{1/\gamma}} w$$

### Marshallian demands for the goods

- Demands for each of the goods is then:

$$\begin{aligned} x_{kt} &= \left( \frac{\alpha_k}{\frac{p_{kt}}{P_t}} \right) c_t \\ &= \left( \frac{\alpha_k}{\frac{p_{kt}}{P_t}} \right) \frac{R(\beta R)^{t/\gamma}}{R + (\beta R)^{1/\gamma}} w \end{aligned}$$

- Notice that demand for goods depends only on preference parameters ( $\alpha_k$ ) and real variables (wealth  $w$ , interest rates  $r$ , relative prices  $p_{kt}/P_t$ )

### Modeling implications

- If utility is time-separable, we can split the problem of choosing  $n$  goods over  $T$  periods into  $T + 1$  problems:
  - decide how much to spend in each of the  $T$  periods (inter-temporal allocation); and
  - take each period budget and decide how to spend it into the  $n$  goods (intra-temporal allocation)
- If intra-temporal preference is CES, we can interpret the indirect utilities of the intra-temporal allocations as *real composite consumption good*.
- From now on, in our macro models we will analyze dynamic consumption behavior assuming that there exist such real composite consumption good.
- We will simply call it the consumption good.

## 4 Intertemporal consumption with uncertainty

### Intertemporal consumption with uncertainty

- Representative consumer lives 2 periods.
- She can save and borrow at interest rate  $r$ .

- Her initial asset is  $a_0$ .
- She doesn't leave any debt or inheritance ( $a_2 = 0$ ).
- Her income  $y_t \geq 0$  in period  $t = 0, 1$ :
  - $y_0$  is known at time of deciding  $c_0$ .
  - $\tilde{y}_1$  is uncertain. It takes value  $y_{1s}$  with probability  $\pi_s$ , depending on the state of nature  $s = 1, \dots, S$ .
  - Notice that  $\sum_{s=1}^S \pi_s = 1$ .
- Her *expected* future income is then

$$\mathbb{E} \tilde{y}_1 = \sum_{s=1}^S \pi_s y_{1s}$$

### Budget constraint

- Budget constraints:

$$\begin{aligned} a_1 &= R(a_0 + y_0 - c_0) \\ a_2 &= R(a_1 + \tilde{y}_1 - \tilde{c}_1) = 0 \end{aligned}$$

- $a_0$  and  $y_0$  are certain (she already have them in her bank).
- $c_0$  and  $a_1$  are certain (she knows what she is choosing *now*).
- $c_1$  is uncertain because she needs to adjust future consumption to income shocks:

$$\begin{aligned} \tilde{c}_1 &= a_1 + \tilde{y}_1 \quad \Rightarrow \\ \mathbb{E} \tilde{c}_1 &= a_1 + \mathbb{E} \tilde{y}_1 \quad \Rightarrow \\ \tilde{c}_1 &= \mathbb{E} \tilde{c}_1 + \underbrace{\tilde{y}_1 - \mathbb{E} \tilde{y}_1}_{\text{forecast error}} \end{aligned}$$

### Consumption plans, contingent on income

State	$\mathbb{P}$	Period 0	Period 1
$s$	$\pi_s$	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_{1s} = a_1 + y_{1s}$

#### Example 5: Only two states of nature

State	Probability	Period 0	Period 1
$L$	$\pi_L$	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_1^L = a_1 + y_1^L$
$H$	$\pi_H$	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_1^H = a_1 + y_1^H$



Consumer wants to maximize her discounted expected utility:

$$\begin{aligned}
 U(c_0, c_1^L, c_1^H, \pi_L, \pi_H) &= \mathbb{E}_{\tilde{y}_2} [u(c_0) + \beta u(c_1)] \\
 &= \pi_L [u(c_0) + \beta u(c_1^L)] + \pi_H [u(c_0) + \beta u(c_1^H)] \\
 &= (\pi_L + \pi_H)u(c_0) + \beta [\pi_L u(c_1^L) + \pi_H u(c_1^H)] \\
 &= u(c_0) + \beta \mathbb{E} u(c_1)
 \end{aligned}$$

$$\begin{aligned}
 U &= \{u(c_0) + \beta \mathbb{E} u(c_1)\} \\
 &= \{u(c_0) + \beta [\pi_L u(c_1^L) + \pi_H u(c_1^H)]\} \\
 &= \{u(a_0 + y_0 - \frac{a_1}{R}) + \beta [\pi_L u(a_1 + y_1^L) + \pi_H u(a_1 + y_1^H)]\}
 \end{aligned}$$

Objective now depends on  $a_1$  alone. Take FOC:

$$\begin{aligned}
 0 &= -\frac{1}{R}u'(c_0) + \beta\pi_L u'(c_1^L) + \beta\pi_H u'(c_1^H) \\
 u'(c_0) &= \beta R [\pi_L u'(c_1^L) + \pi_H u'(c_1^H)] \\
 &= \beta R \mathbb{E} [u'(c_1)] \qquad \qquad \qquad (Euler\ equation)
 \end{aligned}$$

### Wealth and permanent income

- Combining the budget constraints she gets

$$c_0 + \frac{\tilde{c}_1}{R} = \underbrace{a_0 + y_0 + \frac{\tilde{y}_1}{R}}_{\text{wealth } \tilde{W}_0} \qquad \text{(for any possible state of nature)}$$

- Her wealth at time 0 is uncertain because future income is random. But she can form an expectation:

$$c_0 + \frac{\mathbb{E} \tilde{c}_1}{R} = a_0 + y_0 + \frac{\mathbb{E} \tilde{y}_1}{R} = \mathbb{E} \tilde{W}_0$$

- Her *permanent income*  $y_p$  is the constant level of consumption that she *expects* to be able to afford, given her *expected wealth*. Then

$$y_p = \frac{R}{1+R} \mathbb{E} \tilde{W}$$

### The consumer's problem

- She wants to maximize her discounted expected utility (von Neumann-Morgenstern):

$$\begin{aligned}
 U(c_0, \{c_{1s}; \pi_s\}_{s=1}^S) &= \mathbb{E}_{\tilde{y}_2} [u(c_0) + \beta u(c_1)] \\
 &= u(c_0) + \beta \mathbb{E} u(c_1)
 \end{aligned}$$

- subject to contingent plans

$$c_0 + \frac{c_{1s}}{R} = a_0 + y_0 + \frac{y_{1s}}{R} \equiv W_s \quad (\text{for } s = 1, \dots, S)$$

- There are  $S$  constraints (one per state of nature).
- Let  $\lambda_s \pi_s$  be the Lagrange multiplier associated with the  $s^{\text{th}}$  constraint.

### Solving the problem

- The Lagrangian is

$$\begin{aligned} \mathcal{L} &= u(c_0) + \beta \mathbb{E} u(c_1) + \sum_s \lambda_s \pi_s \left( W_s - c_0 - \frac{c_{1s}}{R} \right) \\ &= u(c_0) + \sum_s \pi_s \left[ \beta u(c_{1s}) + \lambda_s \left( W_s - c_0 - \frac{c_{1s}}{R} \right) \right] \end{aligned}$$

- FOCs:

$$\begin{aligned} (\text{wrt } c_0) \quad 0 &= u'(c_0) - \sum_s \pi_s \lambda_s && \Rightarrow u'(c_0) = \mathbb{E} \lambda \\ (\text{wrt } c_{1s}) \quad 0 &= \pi_s \left[ \beta u'(c_{1s}) - \frac{\lambda_s}{R} \right] && \Rightarrow \pi_s \beta R u'(c_{1s}) = \pi_s \lambda_s \end{aligned}$$

### The Euler equation

- Adding up the FOCs wrt  $c_{1s}$ , we get

$$\begin{aligned} \sum_s \pi_s \beta R u'(c_{1s}) &= \sum_s \pi_s \lambda_s \\ \beta R \mathbb{E} u'(c_1) &= \mathbb{E} \lambda \end{aligned}$$

- Substituting  $\mathbb{E} \lambda$  from the first FOC to get

Euler equation

$$u'(c_0) = \beta R \mathbb{E} u'(c_1)$$

#### Side note: Some math worth remembering

- Let  $u$  and  $v$  be functions,  $X$  and  $Z$  random variables, and  $a$  and  $b$  scalars.
- Suppose that  $X$  and  $Z$  depend on parameter  $t$ .
- Then, under fairly general conditions:

$$\mathbb{E} [au(X) + bv(Z)] = a \mathbb{E} u(X) + b \mathbb{E} v(Z)$$

$$\frac{\partial \mathbb{E} u(X)}{\partial t} = \mathbb{E} \left[ u'(X) \frac{\partial X}{\partial t} \right]$$

### A faster way to get the Euler equation

- Instead of having one constraint for each state of nature, just write one: the expected values of the constraint:

$$c_0 + \frac{\mathbb{E} \tilde{c}_1}{R} = \mathbb{E} \tilde{W}_0$$

- Just keep in mind that this is a shortcut: the budget constraint must be satisfied *in every state of nature*, not only in expected values.
- Besides, the consumer is choosing future consumption contingent on each state of nature. She is not just choosing her expected future consumption.

### Solving the problem

- Lagrangian is

$$\mathcal{L} = u'(c_0) + \beta \mathbb{E} u'(c_1) + \lambda \left( \mathbb{E} \tilde{W} - c_0 - \frac{\mathbb{E} c_1}{R} \right)$$

- FOCs

$$\begin{aligned} \text{(wrt } c_0) \quad 0 &= u'(c_0) - \lambda && \Rightarrow u'(c_0) = \lambda \\ \text{(wrt } c_1) \quad 0 &= \beta \mathbb{E} u'(c_1) - \frac{\lambda}{R} && \Rightarrow \beta R \mathbb{E} u'(c_1) = \lambda \end{aligned}$$

### Euler equation, again

- Then, from the two FOCs

$$u'(c_0) = \beta R \mathbb{E} u'(c_1) \quad \text{(Euler equation)}$$

- Euler equation can be written as:

$$\frac{u'(c_0)}{\beta \mathbb{E} u'(c_1)} = R$$

MRS of present consumption for  
future consumption

price of present consumption in  
terms of future consumption

### Example 6: Hall 1978

- Assume that utility is quadratic  $u(c) = \alpha c - 0.5c^2$  and that  $\beta R = 1$ .
- Euler equation is:

$$\mathbb{E} c_1 = c_0$$

- This means that consumption would follow a *random walk*.
- In such case, under the pure life cycle-permanent income hypothesis, a forecast of future consumption obtained by extrapolating today's level by the historical trend is impossible to improve.

### Example 7: CRRA utility, with uncertainty

- Now assume that consumer has constant relative risk aversion:  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ , with  $\sigma > 0$ .

- Euler equation is:

$$c_0^{-\sigma} = \beta R \mathbb{E}(c_1^{-\sigma})$$

- But notice that  $\mathbb{E}(c_1^{-\sigma}) \neq (\mathbb{E} c_1)^{-\sigma}$ , so we can not simply use budget constraint

$$c_0 + \frac{\mathbb{E} \tilde{c}_1}{R} = \mathbb{E} \tilde{W}_0$$

to solve for  $c_0$  and  $\mathbb{E} c_1$ .

- So, in dynamic models with uncertainty, it is often necessary to use numerical methods to analyze the solution of the model.

## References

- Hall, Robert E. (1978). “Stochastic Implications of the Life Cycle-Permanent Income Hypothesis. Theory and Evidence”. In: *The Journal of Political Economy* 86.6, pp. 971–987.
- Varian, Hal R. (1992). *Microeconomic Analysis*. 3rd ed.
- (2014). *Intermediate Microeconomics. With Calculus*. 1st ed.
- Williamson, Stephen D. (2014). *Macroeconomics*. 5th ed. Pearson.