Lecture 12

Asset pricing model

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Table of contents

Contents

1 Introduction

Objective

- Asset pricing theory tries to explain why some assets pay higher average returns than others.
- Accordingly, the objective is to understand the prices or values of claims to uncertain payments.

Risk versus return

- The central aspect is the *risk-return tradeoff*.
- It is rational that investors demand additional return for an asset incorporating more risk.
- Problems arise, however, when one tries to determine the relevant risk factors and their expected compensation.

Theory beginnings

- The basis for this theory was already laid in the 1950s and 60s with the portfolio selection theory by Markowitz and the Capital Asset Pricing Model (CAPM) by Sharpe, for which he received a Nobel Prize in 1990.
- The CAPM significantly shaped and changed financial management.

Current relevance

- Today it is still widely used in practice and plays the centerpiece in the theoretical discussion of asset pricing, although it continues to be sharply criticized.
- This leads to a variety of adaptations and further developments of the CAPM, but so far no model has been able to sufficiently persuade financial scientists and practitioners.
- In this lecture, we derive one of such adaptations: the *consumption-based capital asset pricing model*.

2 The model

The model

- In this lecture we consider an endowment economy where there are many assets.
- The model we develop is similar to the "Consumption and financial assets" model that we studied in lecture 10.
- In addition to having a single risk-free asset, we now assume that there are N risky assets

Future dividends

- For asset $i \in \{0, 1, \ldots, N\}$ and time t we have:

  y_{it} dividend of asset

  P_{it} price of asset (ex-dividend)

  S_{it} number of shares

- Notice that the asset is priced *after* it has paid dividends to its current owner.

Stochastic process of dividends

- Define the vector $y_t \equiv \{y_{0t}, y_{1t}, \ldots, y_{Nt}\}'$
- Let x_t be an informative vector variable that helps predicting future dividends.
- Let $z_t \equiv (y_t, x_t)$.
- We assume that z_t follows a Markov process:

$$F_t(z_{t+1}) = F(z_{t+1} \mid z_t)$$

The consumer

- Infinite horizon
- Instant utility $u(c_t)$ depends on current consumption
- Constant utility discount rate $\beta \in (0, 1)$
- Lifetime expected utility is:

$$\mathbb{E}\left[\sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s\right) \; | z_t\right]$$

Assets and liquidity

In this model, the consumer:

- is endowed with $\{s_{0t}, s_{1t}, \dots, s_{Nt}\}'$ shares.
- Each share of asset i entitles the consumer to receive y_{it} in dividends, and can be sold for P_{it}.
- Then, the total liquidity a_t available to the consumer at t is

$$a_t \equiv \sum_{i=0}^{N} (y_{it} + P_{it})\, s_{it}$$

The budget constrain

- Consumer uses liquidity a_t to pay for consumption c_t and to buy new shares s_{it+1}:

$$c_t + \sum_{i=0}^{N} P_{it} s_{it+1} \leq a_t$$

- Using the definition of a_t, notice that this can be written as

$$c_t + \sum_{i=0}^{N} P_{it}\,(s_{it+1} - s_{it}) \leq \sum_{i=0}^{N} s_{it} y_{it}$$

  which states that total income (rhs) must be enough to pay for consumption and the accumulation of shares (lhs).

- In what follows, we assume this restriction to be binding.

The consumer problem

The consumer problem consist of finding:

- a consumption rule $c_t = c(a_t, z_t)$
- a portfolio rule $s_{i,t+1} = s_i(a_t, z_t) \quad \forall i = \{0, 1, \dots, N\}$

to maximize expected utility:

$$\max_{c_t, s_{0:N,t}} \mathbb{E}\left[\sum_{s=t}^{\infty} \beta^{s-t} u\,(c_s)\ |z_t\right] \quad \text{subject to}$$

$$c_t + \sum_{i=0}^{N} P_{it} s_{it+1} = \sum_{i=0}^{N} (y_{it} + P_{it})\, s_{it} \equiv a_t$$

The Bellman equation

$$V\,(a_t, z_t) = \max\left\{u(c_t) + \beta\,\mathbb{E}\left[V(a_{t+1}, z_{t+1}) \mid z_t\right]\right\}$$

Use the budget constraint to substitute c_t and the definition of liquidity to substitute a_{t+1} :

$$\begin{aligned} V\,(a_t, z_t) = \max_{s_{i,t+1}} \Bigg\{ u\left(a_t - \sum_{i=0}^{N} P_{it} s_{it+1}\right) + \dots \\ \beta\,\mathbb{E}\left[V\left(\sum_{i=0}^{N} (y_{i,t+1} + P_{i,t+1})\, s_{i,t+1},\ z_{t+1}\right) \mid z_t\right]\Bigg\} \end{aligned}$$

Solving the consumer problem

- Taking derivative with respect to s_{it+1}:

$$-P_{it}u'(c_t) + \beta\,\mathbb{E}\left[\frac{\partial V(a_{t+1}, z_{t+1})}{\partial a_{t+1}}(y_{i,t+1} + P_{i,t+1}) \mid z_t\right] = 0$$

- The envelope condition is

$$\frac{\partial V(a_t, z_t)}{\partial a_t} = u'(c_t)$$

- Thus, the Euler equation is:

$$P_{it}u'(c_t) = \beta\,\mathbb{E}\left[u'(c_{t+1})(y_{i,t+1} + P_{i,t+1}) \mid z_t\right]$$

or equivalently

$$u'(c_t) = \beta\,\mathbb{E}\left[\frac{y_{i,t+1} + P_{i,t+1}}{P_{it}}\,u'(c_{t+1}) \mid z_t\right]$$

Side note: Conditional expectation

- The expression

$$\mathbb{E}_t[X_{t+1}] \equiv \mathbb{E}[X_{t+1}|z_t]$$

denotes the expected value of X_{t+1} based on information z_t up to time t.

Side note: The law of iterated expectations

- For two arbitrary random variables y and z, the *law of iterated expectations* says that

$$\mathbb{E}(y) = \mathbb{E}[\mathbb{E}(y|z)]$$

- In words, the unconditional expectation of the conditional expectation of y conditional on z is equal to the unconditional expectation of y.
- This has the following implication for a time series:

$$\mathbb{E}_t[\mathbb{E}_{t+1}(x_{t+2})] = \mathbb{E}_t(x_{t+2})$$

- In other words, your current best guess of your best guess next period of the realization of x two periods from now is equal to your current best guess of x two periods from now.

Example 1: A risk neutral consumer

Assume that consumer is risk neutral:

$$u(c) = c \qquad\qquad u'(c) = 1$$

Then the price of a share of asset i is

$$P_{it} = \beta\,\mathbb{E}\left[y_{i,t+1} + P_{i,t+1} \mid z_t\right]$$

or equivalently

$$P_{it} = \beta\,\mathbb{E}_t\left[y_{i,t+1} + P_{i,t+1}\right]$$

Notice that this implies that

$$P_{it+1} = \beta\,\mathbb{E}_{t+1}\left[y_{i,t+2} + P_{i,t+2}\right]$$

Substitution of the P_{it+1} of the equation for P_{it} results in:

$$\begin{aligned}
P_{it} &= \beta\,\mathbb{E}_t\left[y_{i,t+1} + P_{i,t+1}\right] \\
&= \beta\,\mathbb{E}_t\left[y_{i,t+1} + \beta\,\mathbb{E}_{t+1}\left[y_{i,t+2} + P_{i,t+2}\right]\right] \\
&= \beta\,\mathbb{E}_t\left[y_{i,t+1}\right] + \beta^2\,\mathbb{E}_t\left[\mathbb{E}_{t+1}\left[y_{i,t+2} + P_{i,t+2}\right]\right] \\
&= \beta\,\mathbb{E}_t\left[y_{i,t+1}\right] + \beta^2\,\mathbb{E}_t\left[y_{i,t+2} + P_{i,t+2}\right]
\end{aligned}$$

Last step follows from the Law of Iterated Expectations.

If we keep substituting future prices we get:

$$P_{it} = \mathbb{E}_t\left[\sum_{s=t+1}^{\infty} \beta^{s-t} y_{is}\right] + \lim_{s\to\infty} \mathbb{E}_t\left(\beta^s P_s\right)$$

As long as there is no “bubles”, the limit term is zero.

$$P_{it} = \mathbb{E}_t\left[\sum_{s=t+1}^{\infty} \beta^{s-t} y_{is}\right]$$

This says that [if consumers are risk-neutral] the price of one share of asset i must be equal to the expected discounted value of all future dividends.

The Euler equation

- Define the (random) return on asset i by

$$R_{it} \equiv \frac{y_{it+1} + P_{it+1}}{P_{it}}$$

- Then, we can write the Euler equation as:

$$u'(c_t) = \beta\,\mathbb{E}_t\left[u'\left(c_{t+1}\right) R_{it}\right]$$

- At time t, c_t can be considered non-random. Therefore:

$$1 = \mathbb{E}_t\left[\underbrace{\beta\frac{u'\left(c_{t+1}\right)}{u'(c_t)}}_{\substack{\text{MRS between consumption} \\ \text{in t+1 and t}}} \quad \underbrace{R_{it}}_{\substack{\text{gross rate of return} \\ \text{on asset } i}}\right] \equiv \mathbb{E}_t\left[M_t R_{it}\right]$$

A riskless asset

- Asset 0 is riskless (a discount index bond). Its yields are:

$$y_0 = \left(\overset{t+1}{1}\,,\ \overset{t+2}{0}\,,\ \overset{t+3}{0}\,, \dots \right)$$

- and its future price is $P_{0,t+1} = 0$.
- So its return is

$$R_{0t} \equiv \frac{y_{0t+1} + P_{0t+1}}{P_{0t}} = \frac{1}{P_{0t}}$$

- This implies that:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[M_t R_{0t}\right] = R_{0t}\,\mathbb{E}_t[M_t] \\ \Rightarrow R_{0t} &= \frac{1}{E_t[M_t]} \end{aligned}$$

A formula for risk premium

- Remember that $\text{Cov}[A, B] = \mathbb{E}[AB] - \mathbb{E}[A]\,\mathbb{E}[B]$
- It follows that:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[M_t R_{it}\right] \\ &= \mathbb{E}_t\left[M_t\right]\mathbb{E}_t\left[R_{it}\right] + \text{Cov}_t\left[M_t,\ R_{it}\right] \\ \frac{1}{\mathbb{E}_t\left[M_t\right]} &= \mathbb{E}_t\left[R_{it}\right] + \frac{\text{Cov}_t\left[M_t,\ R_{it}\right]}{\mathbb{E}_t\left[M_t\right]} = R_{0t} \end{aligned}$$

risk premium

$$\underbrace{\mathbb{E}_t\left[R_{it}\right] - R_{0t}}_{\text{risk premium of asset } i} = \frac{-\text{Cov}_t\left[M_t,\ R_{it}\right]}{\mathbb{E}_t\left[M_t\right]}$$

3 Consumption based capital asset pricing model

Market portfolio

- Let's normalize the total number of shares for each asset to $s_i = 1$
- Total market dividend is

$$y_{t+1} = \sum_{i=0}^{N} y_{it+1}$$

- Since this is an endowment economy (no production):

$$c_t = y_t \qquad \forall t$$

A claim to aggregate endowment

- Consider a claim to aggregate endowment next period.

$$y_M = \left(\overset{t+1}{y_{t+1}}, \overset{t+2}{0}, \overset{t+3}{0}, \ldots \right)$$

- If its current price is P_{Mt}, then its return is

$$R_{Mt} = \frac{y_{t+1}}{P_{Mt}}$$

- Its risk premium would be:

$$\mathbb{E}_t \left[R_{Mt} \right] - R_{0t} = \frac{- \operatorname{Cov}_t \left[M_t,\ R_{Mt} \right]}{\mathbb{E}_t \left[M_t \right]}$$

Quadratic utility

- Assume that $u(c) = \alpha c - 0.5 c^2 \qquad \Rightarrow u'(c) = \alpha - c$
- In this case

$$\begin{aligned} M_t \equiv \frac{\beta u'\left(c_{t+1}\right)}{u'(c_t)} &= \frac{\beta \left(\alpha - y_{t+1}\right)}{\alpha - y_t} \\ &= \frac{\beta \alpha}{\alpha - y_t} - \frac{\beta y_{t+1}}{\alpha - y_t} \\ &= \frac{\beta \alpha}{\alpha - y_t} - \frac{\beta P_{Mt}}{\alpha - y_t} R_{Mt} \\ &\equiv \phi_1 + \phi_2 R_{Mt} \end{aligned}$$

- With quadratic utility, the MRS of future consumption for present consumption is an affine function of the market return.

Comparing risk premia

- The ratio of the risk premium of asset i to market risk premium is:

$$\begin{aligned} \frac{\mathbb{E}_t \left[R_{it} \right] - R_{0t}}{\mathbb{E}_t \left[R_{Mt} \right] - R_{0t}} &= \frac{\operatorname{Cov}_t \left[M_t,\ R_{it} \right]}{\operatorname{Cov}_t \left[M_t,\ R_{Mt} \right]} \\ &= \frac{\operatorname{Cov}_t \left[\phi_1 + \phi_2 R_{Mt},\ R_{it} \right]}{\operatorname{Cov}_t \left[\phi_1 + \phi_2 R_{Mt},\ R_{Mt} \right]} \\ &= \frac{\phi_2 \operatorname{Cov}_t \left[R_{Mt},\ R_{it} \right]}{\phi_2 \operatorname{Cov}_t \left[R_{Mt},\ R_{Mt} \right]} \\ &= \frac{\operatorname{Cov}_t \left[R_{Mt},\ R_{it} \right]}{\operatorname{Var}_t \left[R_{Mt} \right]} \\ &\equiv \beta_{it} \end{aligned}$$

- The β_{it} term is a measure of the comovement of the return of asset i with that of the market.

Consumption-based CAPM

- From last expression, we get the consumption-based capital asset pricing model:

$$\text{CCAPM} \qquad \underbrace{\mathbb{E}_t\left[R_{it}\right] - R_{0t}}_{\text{risk premium of asset } i} = \frac{\mathrm{Cov}_t\left[R_{Mt},\ R_{it}\right]}{\mathrm{Var}_t\left[R_{Mt}\right]} \underbrace{\left[\mathbb{E}_t\left[R_{Mt}\right] - R_{0t}\right]}_{\text{market risk premium}}$$

- That is, [if consumers have quadratic utility] the risk premium of asset i is proportional to the market risk premium, where the proportionality factor is given by the β_{it} coefficient.