Lecture 8

Applications of consumer theory

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EC3201 - Teoría Macroeconómica 2

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The Work-Leisure Decision

The setup

- There are only two goods: consumption goods C and time.
- Barter economy: consumer exchanges work time for consumption good.
  - Price of consumption is 1.
  - One hour of work is worth w units of consumption.
- Consumer is endowed with h hours, to be used in:

  leisure: $l =$ time used at home

  work: $N^s =$ time exchanged in the market (labor time)
- The time constraint for the consumer is then

$$l + N^s = h$$

  which states that leisure time plus time spent working must sum to total time available.

The consumer's real disposable income

- For his work, consumer gets $wN^s = w(h - l)$ units of consumption good.
- Consumer also receives π units of consumption good, in the form of real dividend income.
- Consumer must pay a lump-sum tax amount T to the government.
- Therefore, the budget constraint is

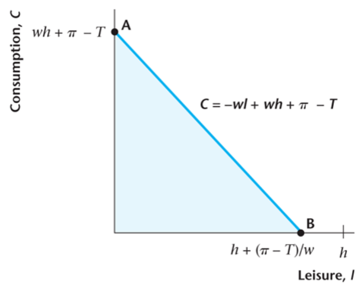
$$C = w(h - l) + \pi - T$$

- which can also be written as

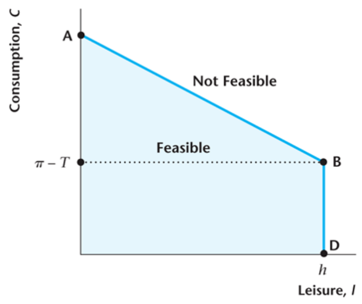
$$C + wl = wh + \pi - T$$

The budget constraint

When $T > \pi$



When $T < \pi$



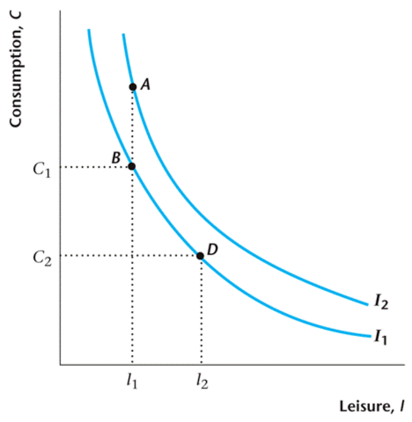
The consumer's preferences

The representative consumer's preferences are defined by

$$U(C, l)$$

with $U(\cdot, \cdot)$ a function that is:

- increasing in both arguments,
- strictly quasiconcave, and
- twice differentiable.



The consumer's problem

- The consumer's optimization problem is to choose C and l so as to maximize $U(C, l)$ subject to his or her budget constraint—that is,

$$\max_{C,l} U(C, l) \qquad \text{s.t.} \begin{cases} C = w(h-l) + \pi - T \\ l \leq h \end{cases}$$

- This problem is a constrained optimization problem, with the associated Lagrangian

$$\mathcal{L} = U(C, l) + \lambda[w(h-l) + \pi - T - C] + \mu(h-l)$$

where λ and μ are the Lagrange multipliers.

Solving the problem

- We assume that there is an interior solution to the consumer's problem where $C > 0$ and $0 < l$.
- This can be guaranteed by assuming that

$$U_C(0, l) = \infty \qquad \text{and} \qquad U_l(C, 0) = \infty$$

- The first-order conditions are

$$U_C(C, l) - \lambda = 0$$
$$U_l(C, l) - \lambda w - \mu = 0$$
$$w(h - l) + \pi - T - C = 0.$$

- Slackness conditions:

$$\mu \geq 0 \qquad h - l \geq 0 \qquad \mu(h - l) = 0$$

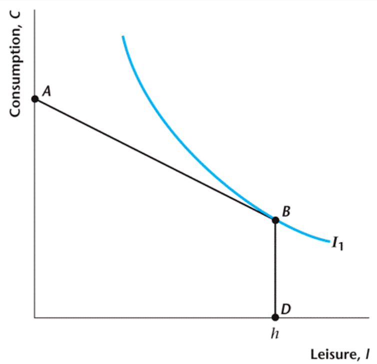
Case 1: $l = h$ (consumer does not work!)

- For this case to be feasible, we require that $C = \pi - T > 0$.
- From the first two FOCs and nonnegativity of multiplier:

$$U_l(\pi - T, h) - wU_C(\pi - T, h) = \mu \geq 0$$

$$\Leftrightarrow w \leq \frac{U_l(\pi - T, h)}{U_C(\pi - T, h)}$$

- Thus, consumer does not work if he has $\pi - T > 0$, and at bundle $(\pi - T, h)$ the market wage rate is less than his MRS of leisure for consumption.
- In a competitive equilibrium we cannot have $l = h$, as this would imply that nothing would be produced and $C = 0$.

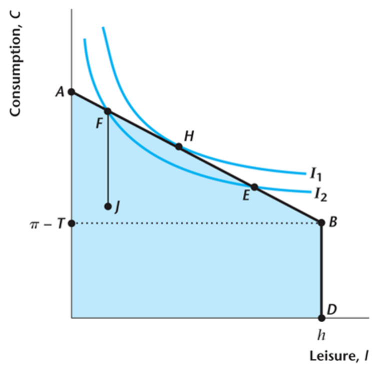


Case 2: $\mu = 0$ (consumer goes to work!)

- From the first two FOCs:

$$U_l(C^*, l^*) = wU_C(C^*, l^*)$$
$$\Leftrightarrow w = \frac{U_l(C^*, l^*)}{U_C(C^*, l^*)}$$

- Thus, consumer works $N^{s^*} = h - l^*$ hours and consumes $C^* = w(h - l^*) + \pi - T$.
- At this allocation, his MRS of leisure for consumption equals the market wage rate.



A parametric example

$U(C,l) = \ln(c) + \gamma \ln(l)$

- FOC

$$\text{MRS}_{lC} = \frac{U_l}{U_C} = \frac{\frac{\gamma}{l}}{\frac{1}{C}} = \frac{\gamma C}{l} = w$$

- Time and budget constraints:

$$w = \frac{\gamma C}{h - N^s}$$

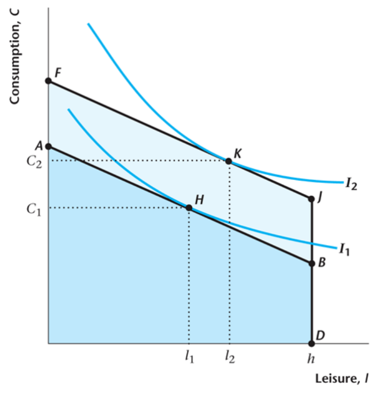
$$C = wN^s + \pi - T$$

- Then

$$N^{s*} = \frac{wh - \gamma(\pi - T)}{(1+\gamma)w}$$

Real Dividends or Taxes Change for the Consumer

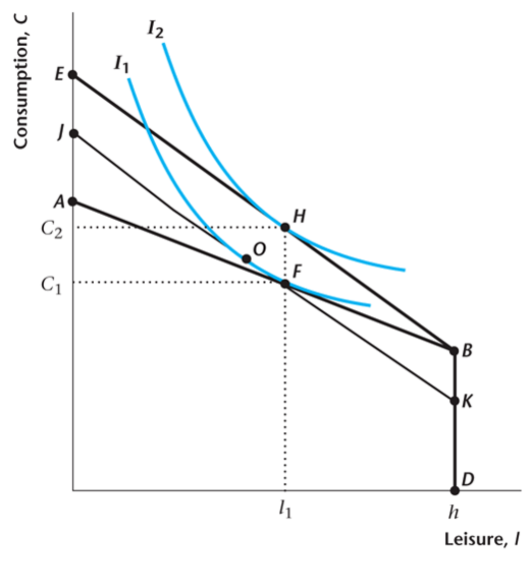
- Assume that consumption and leisure are both normal goods.
- An increase in dividends or a decrease in taxes will then cause the consumer to increase consumption and reduce the quantity of labor supplied (increase leisure).



An Increase in the Market Real Wage Rate

- This has income and substitution effects.
- Substitution effect: the price of leisure rises, so the consumer substitutes from leisure to consumption.
- Income effect: the consumer is effectively more wealthy and, since both goods are normal, consumption increases and leisure increases.
- Conclusion: Consumption must rise, but leisure may rise or fall.

Increase in the Real Wage Rate–Income and Substitution Effects



The labor supply function

- Suppose $l(w)$ is a function that tells us how much leisure the consumer wishes to consume, given the real wage w.
- Then, the labor supply curve is given by

$$N^s(w) = h - l(w)$$

- We do not know whether labor supply is increasing or decreasing in the real wage, because the effect of a wage increase on the consumer's leisure choice is ambiguous.
- Assuming that the substitution effect is larger than the income effect of a change in the real wage, labor supply increases with an increase in the real wage, and the labor supply schedule is upward-sloping.

The slope of the labor supply function

- We do not know whether labor supply is increasing or decreasing in the real wage, because the effect of a wage increase on the consumer's leisure choice is ambiguous.
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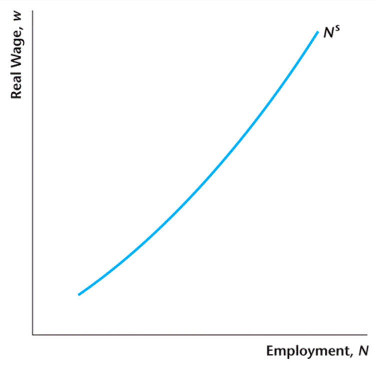
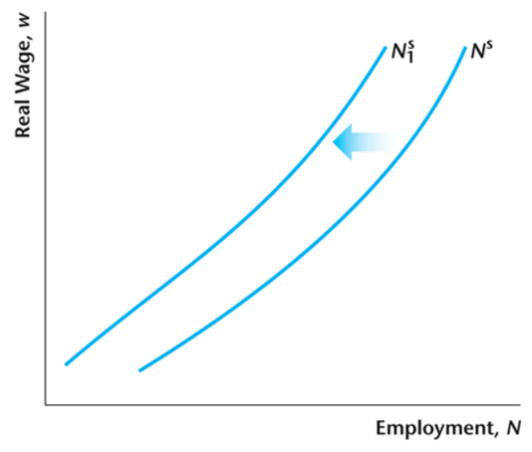


Labor supply response to an increase in dividend

- An increase in nonwage disposable income shifts the labor supply curve to the left, that is, from N^s to N_1^s, because leisure is a normal good



The Work-Leisure Decision: Comparative statics in leisure-consumption model

The economist's problem

- You have a model with n endogenous variables $\mathbf{y}$ and m exogenous variables $\mathbf{x}$, whose solution is described by $\mathbf{y} = \Psi(\mathbf{x})$.
- You have found n model conditions of the form $g(\mathbf{x}, \mathbf{y}) = 0$.
- Problem: How to analyze the comparative statics of the model without an explicit formula for $\Psi(\mathbf{x})$?
- Solution: compute the total derivative of g, using the chain rule.

Side note: The gradient and the Hessian matrix

Let f be a function, $f : \mathbb{R}^n \to \mathbb{R}$, where $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}'$. We denote the first partial derivatives of $f(\mathbf{x})$ by

$$f_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i} \quad \text{and} \quad \nabla f(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

and the Hessian matrix of $f(\mathbf{x})$ by

$$H(\mathbf{x}) = \begin{bmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \dots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \dots & f_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \dots & f_{nn}(\mathbf{x}) \end{bmatrix}$$

Side note: The Jacobian

Let f be a function, $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$:

$$f(\mathbf{x}) = \begin{bmatrix} f^1(\mathbf{x}) \\ \vdots \\ f^m(\mathbf{x}) \end{bmatrix}$$

We denote the Jacobian of $f(\mathbf{x})$ by

$$J(\mathbf{x}) = \begin{bmatrix} f^1_1(\mathbf{x}) & f^1_2(\mathbf{x}) & \dots & f^1_n(\mathbf{x}) \\ f^2_1(\mathbf{x}) & f^2_2(\mathbf{x}) & \dots & f^2_n(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f^m_1(\mathbf{x}) & f^m_2(\mathbf{x}) & \dots & f^m_n(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \nabla f^1(\mathbf{x})' \\ \nabla f^2(\mathbf{x})' \\ \vdots \\ \nabla f^m(\mathbf{x})' \end{bmatrix}$$

Side note: A partitioned Jacobian

- Let $g(\mathbf{x}, \mathbf{y})$ be a function of vectors $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, such that $g : \mathbb{R}^{n+m} \to \mathbb{R}^n$.
- Think of g as a system of n nonlinear equations on n endogenous variables $\mathbf{y}$ and m exogenous variables $\mathbf{x}$.
- The partial Jacobians Dg_y and Dg_x form a partition of the Jacobian:

$$J(\mathbf{x}, \mathbf{y}) = [Dg_y \mid Dg_x] = \begin{bmatrix} g^1_{y_1} & g^1_{y_2} & \cdots & g^1_{y_n} & g^1_{x_1} & g^1_{x_2} & \cdots & g^1_{x_m} \\ g^2_{y_1} & g^2_{y_2} & \cdots & g^2_{y_n} & g^2_{x_1} & g^2_{x_2} & \cdots & g^2_{x_m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ g^n_{y_1} & g^n_{y_2} & \cdots & g^n_{y_n} & g^n_{x_1} & g^n_{x_2} & \cdots & g^n_{x_m} \end{bmatrix}$$

Side note: The total derivative

- The total derivative of $g(\mathbf{x}, \mathbf{y})$ satisfies

$$\sum_{i=1}^{n} \frac{\partial g^k}{\partial y_i} \, \mathrm{d}y_i + \sum_{i=1}^{m} \frac{\partial g^k}{\partial x_i} \, \mathrm{d}x_i = 0, \qquad \forall k = 1, \ldots, n$$

- This can be written in terms of the partitioned Jacobian:

$$0 = \begin{bmatrix} g^1_{y_1} & g^1_{y_2} & \ldots & g^1_{y_n} \\ g^2_{y_1} & g^2_{y_2} & \ldots & g^2_{y_n} \\ \vdots & \vdots & \ddots & \vdots \\ g^n_{y_1} & g^n_{y_2} & \ldots & g^n_{y_n} \end{bmatrix} \begin{bmatrix} \mathrm{d}y_1 \\ \mathrm{d}y_2 \\ \vdots \\ \mathrm{d}y_n \end{bmatrix} + \begin{bmatrix} g^1_{x_1} & g^1_{x_2} & \ldots & g^1_{x_m} \\ g^2_{x_1} & g^2_{x_2} & \ldots & g^2_{x_m} \\ \vdots & \vdots & \ddots & \vdots \\ g^n_{x_1} & g^n_{x_2} & \ldots & g^n_{x_m} \end{bmatrix} \begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \vdots \\ \mathrm{d}x_m \end{bmatrix}$$

$$= Dg_y \, \mathrm{d}\mathbf{y} + Dg_x \, \mathrm{d}\mathbf{x}$$

- Then $\mathrm{d}\mathbf{y} = -[Dg_y]^{-1} Dg_x \, \mathrm{d}\mathbf{x}$, assuming inverse is defined.

Comparative statics in leisure-consumption model

In our leisure-consumption model, the solution required that:

$$g_1(c, l, w, \pi) = U_l - wU_c = 0$$
$$g_2(c, l, w, \pi) = c - wh + wl - \pi = 0$$

Therefore

$$\begin{aligned} 0 &= \begin{bmatrix} g_c^1 & g_l^1 \\ g_c^2 & g_l^2 \end{bmatrix} \begin{bmatrix} \mathrm{d}c \\ \mathrm{d}l \end{bmatrix} + \begin{bmatrix} g_w^1 & g_\pi^1 \\ g_w^2 & g_\pi^2 \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}\pi \end{bmatrix} \\ &= \begin{bmatrix} U_{lc} - wU_{cc} & U_{ll} - wU_{cl} \\ 1 & w \end{bmatrix} \begin{bmatrix} \mathrm{d}c \\ \mathrm{d}l \end{bmatrix} + \begin{bmatrix} -U_c & 0 \\ l - h & -1 \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}\pi \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} \mathrm{d}c \\ \mathrm{d}l \end{bmatrix} = \begin{bmatrix} U_{lc} - wU_{cc} & U_{ll} - wU_{cl} \\ 1 & w \end{bmatrix}^{-1} \begin{bmatrix} U_c & 0 \\ h - l & 1 \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}\pi \end{bmatrix}$$

$$= \frac{1}{\nabla} \begin{bmatrix} w & wU_{cl} - U_{ll} \\ -1 & U_{lc} - wU_{cc} \end{bmatrix} \begin{bmatrix} U_c & 0 \\ h - l & 1 \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}\pi \end{bmatrix}$$

$$= \frac{1}{\nabla} \begin{bmatrix} wU_c + (h-l)(wU_{cl} - U_{ll}) & wU_{cl} - U_{ll} \\ -U_c + (h-l)(U_{lc} - wU_{cc}) & U_{lc} - wU_{cc} \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}\pi \end{bmatrix}$$

where

$$\nabla = -\left(w^2 U_{cc} - 2wU_{cl} + U_{ll}\right) = -\begin{vmatrix} 0 & 1 & w \\ 1 & U_{cc} & U_{cl} \\ w & U_{lc} & U_{ll} \end{vmatrix} \geq 0$$

The comparative statics follows from:

$$\frac{\mathrm{d}c}{\mathrm{d}\pi} = \frac{wU_{cl} - U_{ll}}{\nabla} \quad > 0 \qquad (c \text{ is normal})$$

$$\frac{\mathrm{d}c}{\mathrm{d}w} = \frac{wU_c + (h-l)(wU_{cl} - U_{ll})}{\nabla} \quad > 0$$

$$\frac{\mathrm{d}l}{\mathrm{d}\pi} = \frac{U_{lc} - wU_{cc}}{\nabla} \quad > 0 \qquad (l \text{ is normal})$$

$$\frac{\mathrm{d}l}{\mathrm{d}w} = \frac{-U_c + (h-l)(U_{lc} - wU_{cc})}{\nabla} \quad {\color{red}?} \quad 0$$

Choice under uncertainty

Choice under uncertainty

- Until now, we have been concerned with the behavior of a consumer under conditions of certainty.
- However, many choices made by consumers take place under conditions of uncertainty.
- In this section we explore how the theory of consumer choice can be used to describe such behavior.

The choices

- The first question to ask is what is the basic “thing” that is being chosen?
- The consumer is presumably concerned with the probability distribution of getting different consumption bundles of goods.
- A probability distribution consists of a list of different outcomes—in this case, consumption bundles—and the probability associated with each outcome.
- When a consumer decides how much automobile insurance to buy or how much to invest in the stock market, he is in effect deciding on a pattern of probability distribution across different amounts of consumption.

Contingent consumption

- Let us think of the different outcomes of some random event as being different states of nature.
- A contingent consumption plan is a specification of what will be consumed in each different state of nature.
- Contingent means depending on something not yet certain.
- People have preferences over different plans of consumption, just like they have preferences over actual consumption.
- We can think of preferences as being defined over different consumption plans.

Utility functions and probabilities

- If the consumer has reasonable preferences about consumption in different circumstances, then we can use a utility function to describe these preferences.
- However, uncertainty does add a special structure to the choice problem.
- How a person values consumption in one state as compared to another will depend on the probability that the state in question will actually occur.
- For this reason, we will write the utility function as depending on the probabilities as well as on the consumption levels.

Utility with discrete random outcomes

- If there are n possible states of nature s, then c is a discrete random variable with support $\{c_1, \ldots, c_n\}$, whose values are realized with probabilities $\{p_1, \ldots, p_n\}$.

s	$\mathbb{P}$	c	$u(c)$
1	π_1	c_1	$u(c_1)$
2	π_2	c_2	$u(c_2)$
$\vdots$		$\vdots$	
n	π_n	c_n	$u(v_n)$

- Utility is

$$U(c_1, \ldots, c_n; \pi_1, \ldots, \pi_n) = \sum_{i=1}^{n} \pi_i u(c_i)$$

Utility with continuous random outcomes

- If there are infinite states of nature, we think of c as a continuous random variable.
- If c has support $\mathbf{C}$, pdf $f(c)$ and cdf $F(c)$, then utility is

$$U(c, f) = \int_{\mathbf{c}} f(c)u(c) \, \mathrm{d}c$$

$$= \int_{\mathbf{c}} u(c) \, \mathrm{d}F(c)$$

von Neumann-Morgenstern utility

- We refer to a utility function U with the particular form described here as an expected utility function, or, sometimes, a von Neumann-Morgenstern utility function:

$$U(c, \mathbb{P}) \equiv \mathbb{E}\, u(c) = \begin{cases} \sum_{i=1}^{n} \pi_i u(c_i) & \text{discrete} \\ \int_{\mathbf{c}} u(c)\ \mathrm{d}F(c) & \text{continuous} \end{cases}$$

- We refer to $u(c)$ as the Bernoulli utility function.

Choice under uncertainty: Demand for insurance

Growing potatoes in uncertain weather

- A farmer grows potatoes for own consumption.
- The weather s can be *good* or *bad*, affecting the amount of potatoes (real income y) he actually harvests:

s (weather)	$\mathbb{P}$	y
g (good)	π_g	W
b (bad)	π_b	$W - L$

- That is, if weather is bad, he loses L potatoes.
- Expected consumption of potatoes:

$$\mathbb{E}\, c = \mathbb{E}\, y = (1 - \pi_b)W + \pi_b(W - L) = W - \pi_b L$$

An insurance contract

- Farmer can insure K potatoes, premium is γ per unit.
- Choices are contingent consumption plans:

s	$\mathbb{P}$	y	insure	c
g	π_g	W	$-\gamma K$	$W - \gamma K$
b	π_b	$W - L$	$(1 - \gamma)K$	$W - \gamma K + K - L$

Expected utility of buying insurance coverage K

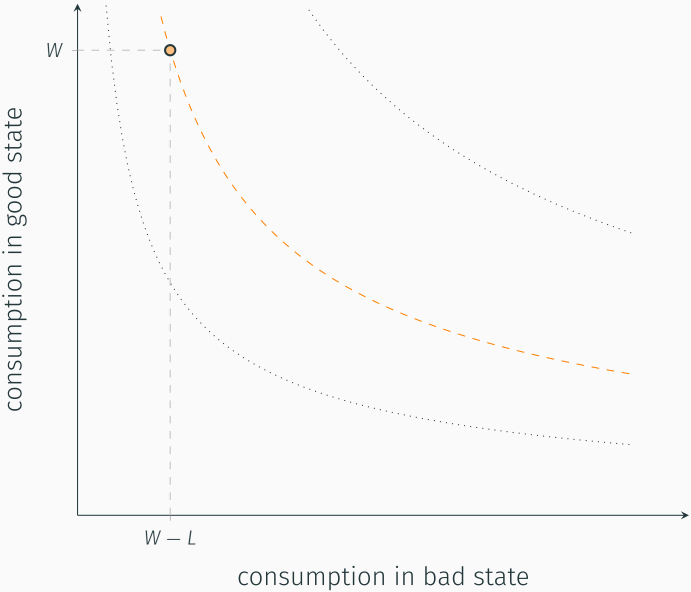
- Expected utility is

$$
\begin{aligned}
U(c_g, c_b; \pi_g, \pi_b) &\equiv \mathbb{E}\, u(c) \\
&= \pi_g u(c_g) + \pi_b u(c_b) \\
&= \pi_g u(W - \gamma K) + \pi_b u(W - \gamma K + K - L)
\end{aligned}
$$

- MRS of bad-weather potatoes for one good-weather potato is

$$
MRS_{bg} = \frac{U_{c_b}}{U_{c_g}} = \frac{\pi_b u'(c_b)}{\pi_g u'(c_g)}
$$

Objective function: $\mathbb{E}\,u(c) = U(c_g, c_b, \pi_g, \pi_b) = \pi_g u(c_g) + \pi_b u(c_b)$



Budget constraint $(c_g, c_b) = (y_g - \gamma K,\ y_b + (1-\gamma)K)$

- We have

$$K = \frac{y_g - c_g}{\gamma} = \frac{c_b - y_b}{1-\gamma}$$

- Therefore

$$c_g + \tfrac{\gamma}{1-\gamma} c_b = y_g + \tfrac{\gamma}{1-\gamma} y_b$$

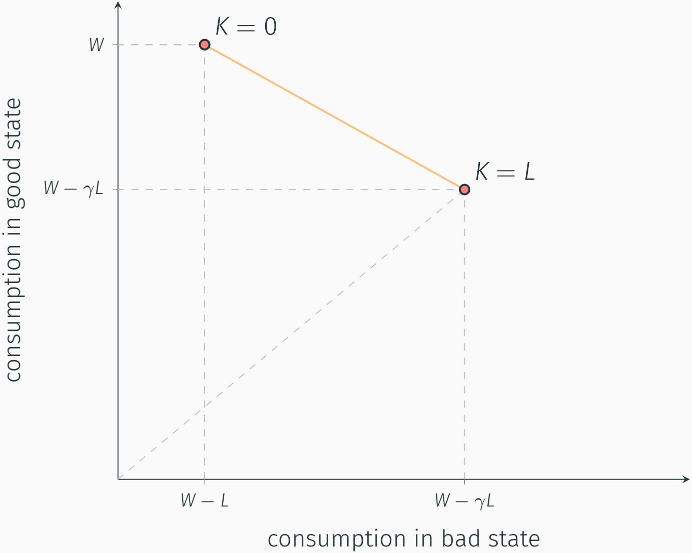
- Substitute $y_g = W$ and $y_b = W - L$ to get

$$\begin{aligned} c_g + \tfrac{\gamma}{1-\gamma} c_b &= W + \tfrac{\gamma}{1-\gamma}(W - L) \\ &= \tfrac{1}{1-\gamma} W - \tfrac{\gamma}{1-\gamma} L \end{aligned}$$

- The relative price (in terms of potatoes in good weather) of a potato in bad weather is $p = \frac{\gamma}{1-\gamma}$

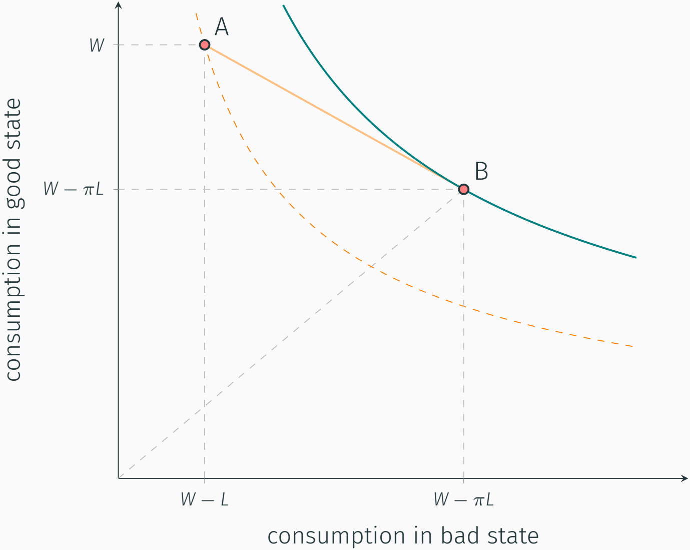
Budget constraint:

$$c_g + \frac{\gamma}{1-\gamma} c_b = \frac{1}{1-\gamma} W - \frac{\gamma}{1-\gamma} L$$



Optimality condition:

$$MRS_{bg} = \frac{\pi_b u'(c_b)}{\pi_g u'(c_g)} = \frac{\gamma}{1-\gamma} = p$$



Demand for insurance

- Of course, we could also solve for optimal K directly:

$$\max_K \{\pi_g u(W - \gamma K) + \pi_b u(W - L - \gamma K + K)\}$$

- FOC:

$$0 = -\gamma \pi_g u'(W - \gamma K) + (1-\gamma)\pi_b u'(W - L - \gamma K + K)$$

$$\Leftrightarrow \qquad \frac{\pi_b u'(W - L - \gamma K + K)}{\pi_g u'(W - \gamma K)} = \frac{\gamma}{1-\gamma}$$

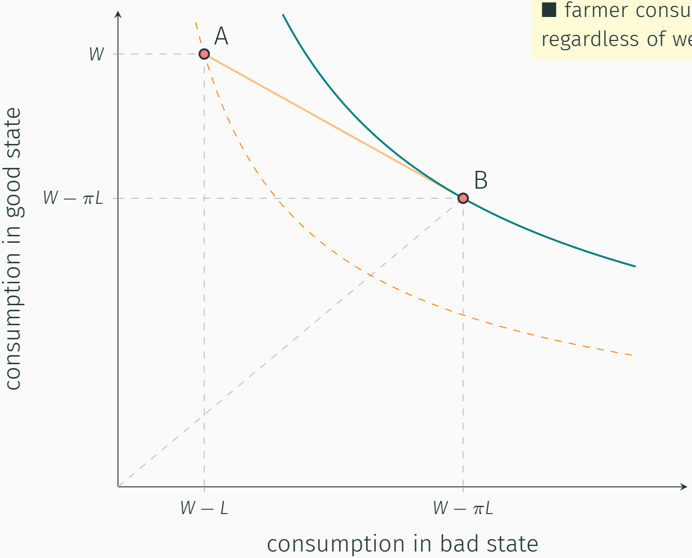
Risk of losses and price of insurance

- The market price of insurance should satisfy $\gamma \geq \pi_b$, so the insurer gets enough revenue γK to cover expected payments $\pi_b K$. This implies that:

$$\begin{aligned}
\gamma &\geq \pi_b \\
1 - \pi_b &\geq 1 - \gamma \\
\gamma(1 - \pi_b) &\geq \pi_b(1 - \gamma) \\
1 &\geq \frac{\pi_b(1-\gamma)}{\gamma(1-\pi_b)} = \frac{u'(c_g)}{u'(c_b)} && \text{(from FOC)} \\
u'(c_b) &\geq u'(c_g) \\
c_b &\leq c_g && \text{(assuming risk aversion)}
\end{aligned}$$

- Consumer gets full insurance iif it's actuarially fair.

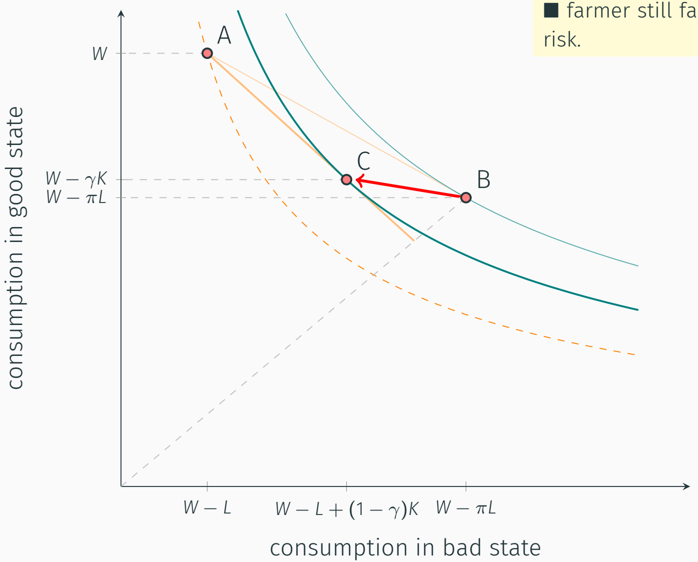
Case $\gamma = \pi_b$: actuarially fair insurance



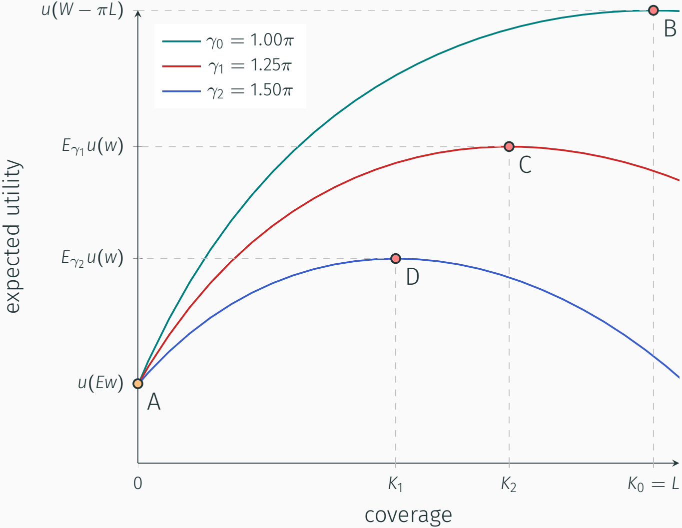
- $K^* = L$: full insurance
- farmer consumes $W - \pi_b L$, regardless of weather.

Case $\gamma > \pi_b$: insurer expects a profit

- $K^* < L$: no full insurance
- farmer still faces (lower) risk.



Increasing the premium: As insurance gets expensive, consumer buys less coverage.



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Example 1:

Logarithmic utility

Let's now assume that $u(c) = \ln(c)$ From the FOC:

$$\gamma \pi_g u'(W - \gamma K) = (1-\gamma)\pi_b u'(W - L + (1-\gamma)K)$$

$$\gamma \pi_g [W - L + (1-\gamma)K] = (1-\gamma)\pi_b (W - \gamma K)$$

$$\pi_g \gamma (W - L) + \pi_g \gamma (1-\gamma) K = \pi_b (1-\gamma) W - \pi_b \gamma (1-\gamma) K$$

$$(\pi_b + \pi_g)(1-\gamma)\gamma K = (\pi_b - \gamma(\pi_b + \pi_g))W + \gamma \pi_g L$$

$$\gamma(1-\gamma)K = (\pi_b - \gamma)W + \gamma(1-\pi_b)L$$

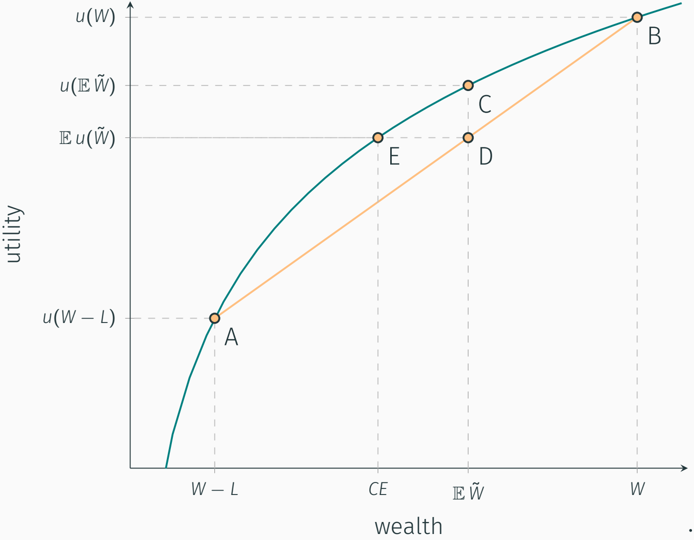
$$K^* = \frac{1-\pi_b}{1-\gamma} L - \frac{\gamma - \pi_b}{\gamma(1-\gamma)} W$$

Optimal contingent consumption plans:

s	$\mathbb{P}$	y	c^*
g	π_g	W	$\frac{1-\pi_b}{1-\gamma}(W - \gamma L)$
b	π_b	$W - L$	$\frac{\pi_b}{\gamma}(W - \gamma L)$

Choice under uncertainty: Risk aversion

A risk averse consumer



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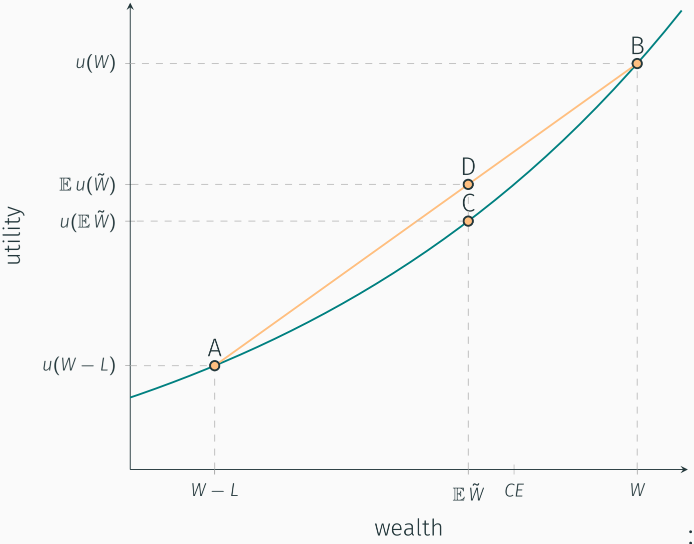
A risk loving consumer



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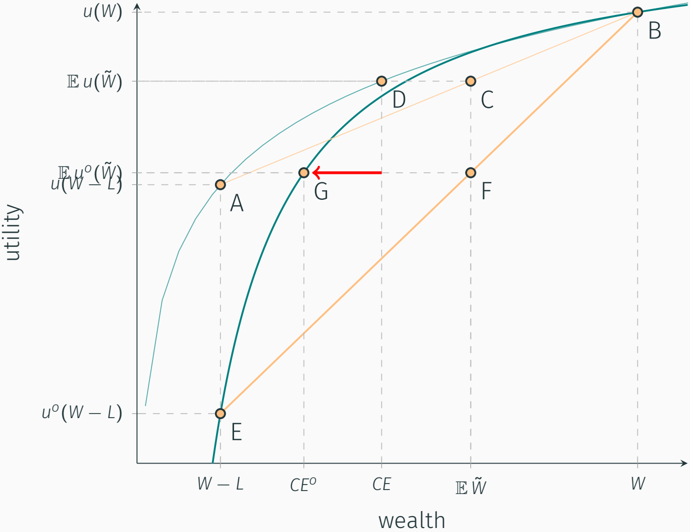
Measuring risk aversion

- A consumer with a von Neumann-Morgenstern utility function can be one of the following:
  - Risk-averse, with a concave utility function;
  - Risk-neutral, with a linear utility function, or;
  - Risk-loving, with a convex utility function.
- Then, the degree of risk-aversion a consumer displays would be related to the curvature of their Bernoulli utility function $u(W)$.
- The more ”curved” a concave $u(W)$ is, the lower will be a consumer’s certainty equivalent, and the higher their risk premium.
- How do we measure the curvature of a function?
- Simple - using the function’s second derivative.

Arrow-Pratt measure of risk aversion

Absolute	Relative
$$\frac{-u''(W)}{u'(W)}$$	$$\frac{-u''(W)W}{u'(W)}$$
CARA	**CRRA**
$$u(c) = -e^{-\rho c}$$	$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

A change in risk aversion



Choice under uncertainty: A risky asset

A risky asset

- Consider a simple two-period portfolio problem involving two assets, one with a risky (gross) return $\tilde{R} \geq 0$ and one with a sure (gross) return $R_f \geq 1$.
- Let w be initial wealth, and let $x \in [0, 1]$ be the share of wealth invested in the risky asset.

s	$\mathbb{P}$	risky	risk-free	c
$\tilde{R} = R$	$f(R)$	xw	$(1-x)w$	$[(1-x)R_f + xR]w$

- In this case the second-period wealth can be written as

$$\begin{aligned} \tilde{W} &= (1-x)R_f w + x\tilde{R}w \\ &= [(1-x)R_f + x\tilde{R}]w \end{aligned}$$

- Note that $\tilde{W}$ is a random variable since $\tilde{R}$ is random.

Expected utility

- The expected utility from investing x in the risky asset:

$$v(x) = \mathbb{E}\, u(c) = \mathbb{E}\, u\left([(1-x)R_f + x\tilde{R}]w\right)$$

- The portfolio problem is then to choose $x \in [0,1]$ to maximize $v(x)$:

$$\mathcal{L}(x,\mu,\lambda) = \mathbb{E}\, u\left([(1-x)R_f + x\tilde{R}]w\right) + \mu x + \lambda(1-x)$$

- Conditions:

$$\mathbb{E}\left\{u'(\tilde{W})(\tilde{R} - R_f)w\right\} + \mu - \lambda = 0$$

$$\begin{aligned} &\mu \geq 0 && x \geq 0 && \mu x = 0 \\ &\lambda \geq 0 && x \leq 1 && \lambda(1-x) = 0 \end{aligned}$$

Second order condition

- Notice that second derivative is

$$\mathbb{E}\left\{u''(\tilde{W})(\tilde{R}-R_f)^2w^2\right\} < 0 \quad \text{iif} \quad u''(\tilde{W}) < 0$$

- SOC requires that consumer is risk-averse.

Slackness conditions

- The slackness conditions (SC) imply:
    - if $x = 0$, 2^{nd} group of SC satisfied with $\lambda = 0$.
    - if $x = 1$, 1^{st} group of SC satisfied with $\mu = 0$.
    - if $0 < x < 1$, both groups of SC satisfied with $\lambda = \mu = 0$.

- Then, we only need to analyze 3 cases:
    - $x = 0 \quad \Rightarrow \quad \mathbb{E}\left\{u'(\tilde{W})(\tilde{R} - R_f)\right\} = -\mu \leq 0$
    - $x = 1 \quad \Rightarrow \quad \mathbb{E}\left\{u'(\tilde{W})(\tilde{R} - R_f)\right\} = \lambda \geq 0$
    - $0 < x < 1 \quad \Rightarrow \quad \mathbb{E}\left\{u'(\tilde{W})(\tilde{R} - R_f)\right\} = 0$

Case 1: $x = 0 \Rightarrow \tilde{W} = wR_f$

$$\mathbb{E}\left\{u'(\tilde{W})(\tilde{R} - R_f)\right\} \leq 0$$

$$\mathbb{E}\left\{u'(\tilde{W})\tilde{R}\right\} \leq \mathbb{E}\left\{u'(\tilde{W})R_f\right\}$$

$$\mathbb{E}\left\{u'(wR_f)\tilde{R}\right\} \leq \mathbb{E}\left\{u'(wR_f)R_f\right\}$$

$$u'(wR_f)\,\mathbb{E}\left\{\tilde{R}\right\} \leq u'(wR_f)R_f$$

$$\mathbb{E}\left\{\tilde{R}\right\} \leq R_f$$

Consumer does not invest in risky asset if its return is lower than the risk-free return.

Case 2: $x = 1 \Rightarrow \tilde{W} = w\tilde{R}$

$$
\begin{aligned}
\mathbb{E}\left\{u'(\tilde{W})(\tilde{R} - R_f)\right\} &\geq 0 \\
\mathbb{E}\left\{u'(\tilde{W})\tilde{R}\right\} &\geq \mathbb{E}\left\{u'(\tilde{W})R_f\right\} \\
\mathbb{E}\left\{u'(w\tilde{R})\tilde{R}\right\} &\geq \mathbb{E}\left\{u'(w\tilde{R})R_f\right\} \\
R_f &\leq \frac{\mathbb{E}\left\{u'(w\tilde{R})\tilde{R}\right\}}{\mathbb{E}\left\{u'(w\tilde{R})\right\}}
\end{aligned}
$$

Consumer does not invest in risk-free asset if its return is ”too low”. We need more details about the $\tilde{R}$ process and utility u to determine what ”too low” is.

Case 3: $0 < x < 1$

$$
\begin{aligned}
0 &= \mathbb{E}\left\{u'(\tilde{W})(\tilde{R} - R_f)\right\} \\
&= \operatorname{Cov}\left[u'(\tilde{W}),\ \tilde{R} - R_f\right] + \mathbb{E}\left[u'(\tilde{W})\right]\mathbb{E}\left[\tilde{R} - R_f\right]
\end{aligned}
$$

Then

$$
\mathbb{E}\,\tilde{R} - R_f = \frac{-\operatorname{Cov}\left[u'(\tilde{W}),\ \tilde{R}\right]}{\mathbb{E}\,u'(\tilde{W})} > 0
$$

Example 2:

”Investing” in ”Tiempos”

- In ”Tiempos” lottery, you pick one number out of 100, all of them with equal probability (1%) of winning.
- In *winning* state, your gross return is $\tilde{R} = 72$.
- If *losing* state, your gross return is $\tilde{R} = 0$.
- If you don't play, you keep your money ($R_f = 1$).
- Expected return on lottery is

$$\mathbb{E}\,\tilde{R} = 0.99 \times 0 + 0.01 \times 72 = 0.7128 < 1 = R_f$$

- Therefore, a risk-averse consumer would never play ”Tiempos”.

Intertemporal consumption

Adding a time dimension

- So far we have only studied static choices.
- Life is full of intertemporal choices: Should I study for my test today or tomorrow? Should I save or should I consume now?
- We will present a simple model: the Life-Cycle/Permanent Income Model of Consumption.
- Developed by Modigliani (Nobel winner 1985) and Friedman (Nobel winner 1976).
- Will allow us to address several key issues: effects of government programs including Social Security, government debts and deficits.

The model

- Representative household lives 2 periods.
- Utility function:

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

  - c_0 is consumption in first (current) period of life,
  - c_1 is consumption in second (future) period of life,
  - $0 < \beta < 1$ measures household's degree of impatience.
- Preferences over c_0, c_1 satisfy monotonicity ($u' > 0$) and convexity ($u'' < 0$).

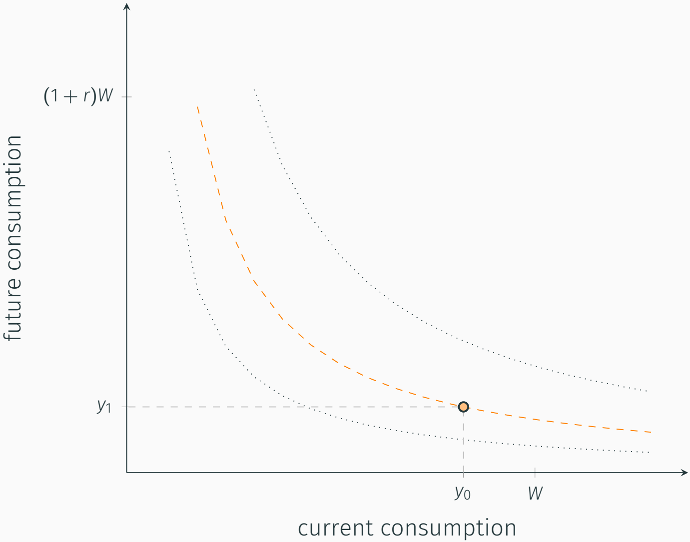
More on preferences

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

- Consumption smoothing motive, partially offset by discounting.
- Assume c_0 and c_1 are normal: more income $\Rightarrow$ more of both.
- Intertemporal marginal rate of substitution measures willingness to substitute consumption over time:

$$MRS_{c_0,c_1} = \frac{U_{c_0}(c_0, c_1)}{U_{c_1}(c_0, c_1)} = \frac{u'(c_0)}{\beta u'(c_1)}$$

$U(c_0, c_1) = u(c_0) + \beta u(c_1)$



Budget constraint I

- Abstract from labor/lesiure tradeoff.
- (Labor) income $y_t \geq 0$ in period $t = 0, 1$.
- Initial wealth $a_0 \geq 0$.
- Consumer can save part of income or initial wealth in the first period, or it can borrow against future income y_1.
- Interest rate on both savings and on loans is equal to r. Gross interest rate $R \equiv 1 + r$
- Let $s_t = y_t - c_t$ denote saving.
- Budget constraint in first period:

$$a_1 = R(a_0 + s_0)$$

- Budget constraint in second period:

$$a_2 = R(a_1 + s_1) = 0$$

Budget constraint (II)

- Combining both constraints:

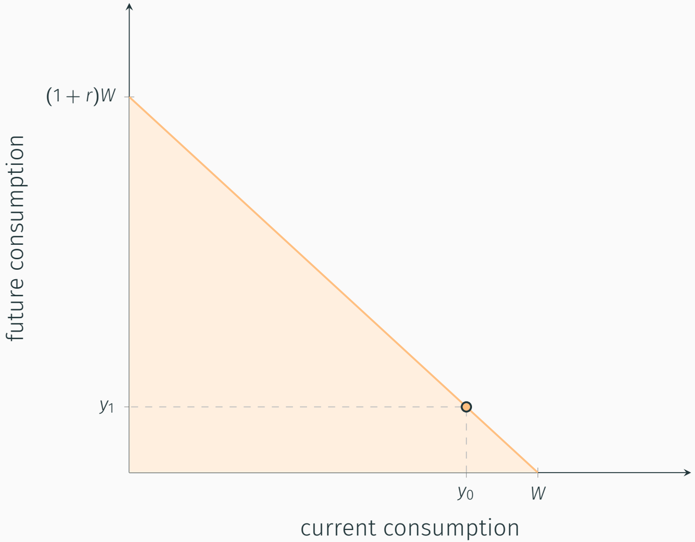
$$R(a_0 + s_0) + s_1 = 0 \qquad \Rightarrow \qquad -s_0 - \frac{s_1}{R} = a_0$$

- Substitute $s_t = y_t - c_t$

$$c_0 + \frac{c_1}{R} = y_0 + \frac{y_1}{R} + a_0 = H + a_0 \equiv W \qquad \text{(PVBC)}$$

- We have normalized the price of the consumption good in the first period to 1.
- Gross interest rate $R \equiv 1 + r$ is the relative price of consumption goods today to consumption goods tomorrow.
- Called the present value budget constraint (PVBC).

$c_0 + \frac{c_1}{R} = W$



The consumer's problem

$$\max_{c_0,c_1} \{u(c_0) + \beta u(c_1)\} \qquad c_0 + \frac{c_1}{R} = W$$

- Form Lagrangian with multiplier $\lambda \geq 0$

$$\mathcal{L}(c_0, c_1, \lambda) = u(c_0) + \beta u(c_1) + \lambda \left(W - c_0 - \frac{c_1}{R} \right)$$

- FOCs:

$$u'(c_0) = \lambda$$

$$\beta u'(c_1) = \frac{\lambda}{R}$$

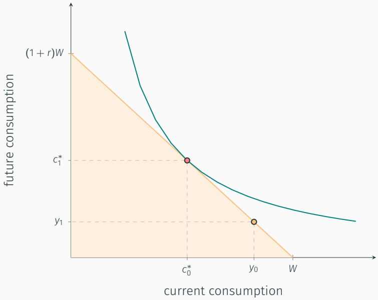
- Combine to get

Euler equation

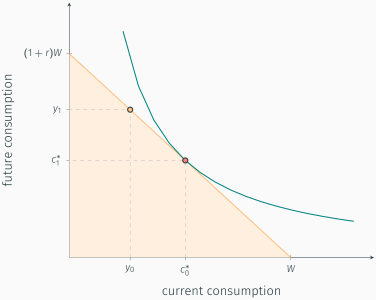
$$u'(c_0) = \beta R u'(c_1)$$

$u'(c_0) = \beta R u'(c_1)$

Consumer is a lender



Consumer is a borrower



Implications of the Euler equation

$$u'(c_0) = \beta R u'(c_1)$$

- Can also be written

$$MRS_{c_0,c_1} = 1 + r$$

- Recall that u is concave, so $u'' < 0 \Rightarrow u'(c)$ is decreasing. So if:
  - $\beta(1+r) > 1 \quad \Rightarrow \quad u'(c_0) > u'(c_1) \quad \Rightarrow \quad c_0 < c_1$
  - $\beta(1+r) < 1 \quad \Rightarrow \quad u'(c_0) < u'(c_1) \quad \Rightarrow \quad c_0 > c_1$
  - $\beta(1+r) = 1 \quad \Rightarrow \quad u'(c_0) = u'(c_1) \quad \Rightarrow \quad c_0 = c_1$
- Behavior of consumption over time depends on rate of time preference relative to interest rate.
- If equal, perfect consumption smoothing.

Example 3:

Logarithmic utility

$u(c) = \ln(c)$

- Euler equation:

$$\frac{1}{c_0} = \frac{\beta R}{c_1} \qquad \Rightarrow \qquad c_1 = \beta R c_0$$

- Using the PVBC

$$c_0 = W - \frac{c_1}{R} = W - \beta c_0$$

- So that

$$c_0 = \tfrac{1}{1+\beta} W \qquad s_0 = \tfrac{1}{1+\beta}\left(\beta y_0 - a_0 - \frac{y_1}{R}\right)$$

$$c_1 = \tfrac{\beta R}{1+\beta} W \qquad a_1 = \tfrac{1}{1+\beta}\left[\beta R(y_0 + a_0) - y_1\right]$$

- Value function:

$$V(W,r) = (1+\beta)\ln W + \beta \ln R + \beta \ln \beta - (1+\beta)\ln(1+\beta)$$

- Increasing wealth W, regardless of source, increases consumer utility:

$$\frac{\partial V}{\partial W} = \frac{1+\beta}{W}$$

- Effect of a change in interest rate r depends on wealth composition, which in turn determines whether the consumer has positive or negative assets a_1 at the end of period 1:

$$\begin{aligned}\frac{\partial V}{\partial r} &= \frac{1}{R^2 W}\left[\beta R(y_0 + a_0) - y_1\right] \\ &= \frac{1+\beta}{R^2 W} a_1\end{aligned}$$

Example 4:

CRRA utility

- The logarithmic utility from last example is just a special case of the constant relative risk aversion(CRRA) utility, when $\sigma = 1$.

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

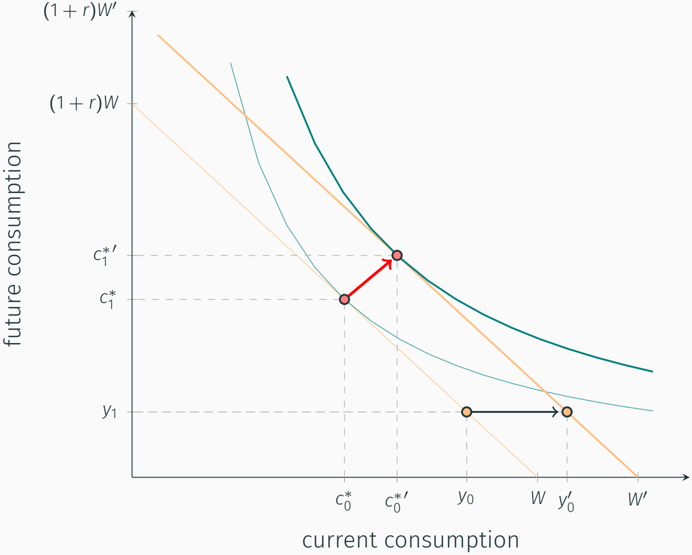
- With CRRA utility, the Bellman equation becomes

$$c_0^{-\sigma} = \beta R c_1^{-\sigma} \qquad \Rightarrow \quad c_1 = (\beta R)^{1/\sigma} c_0$$

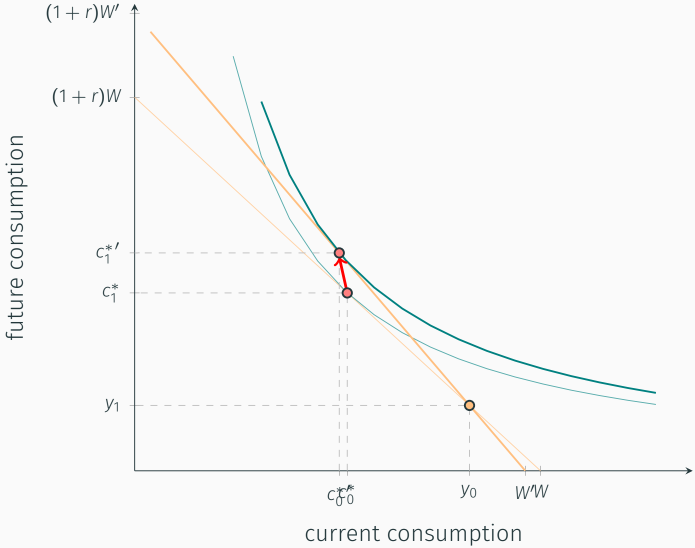
- Use budget constraint $c_0 + \frac{c_1}{R} = W$ to solve for c_0 and c_1:

$$c_0 = \frac{R}{R + (\beta R)^{1/\sigma}} W \qquad c_1 = \frac{R(\beta R)^{1/\sigma}}{R + (\beta R)^{1/\sigma}} W$$

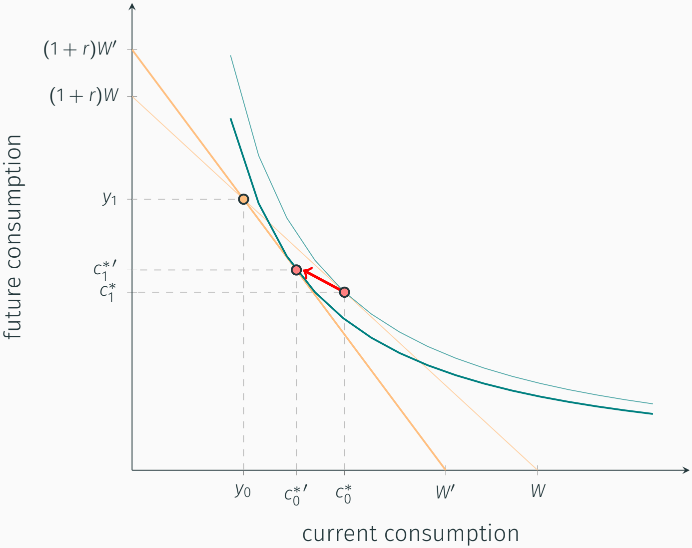
Increasing wealth



Increasing interest rate: lender



Increasing interest rate: borrower



Intertemporal consumption: Many goods, two time periods

The model

- A consumer lives two periods, and chooses among $n+1$ goods in each period: x_{it} for $i \in \{0, 1, \dots, n\}$ and $t \in \{0, 1\}$.
- Utility function depends on $2n+2$ goods:

$$U = \frac{(\alpha_0 x_{00}^{\rho} + \cdots + \alpha_n x_{n0}^{\rho})^{\frac{1-\gamma}{\rho}}}{1-\gamma} + \beta \frac{(\alpha_0 x_{01}^{\rho} + \cdots + \alpha_n x_{n1}^{\rho})^{\frac{1-\gamma}{\rho}}}{1-\gamma}$$

- Let $\mathbf{x}_t$ be the bundle of goods consumed at time t:

$$\mathbf{x}_t = [x_{0t}, x_{1t}, \dots, x_{nt}]$$

Constraints in nominal terms

- Consumer can save and borrow money at nominal interest rate i.
- The budget constraint says that the present value of all consumption purchases must equal the present value of nominal income Y_t:

$$\sum_{k=0}^{n} p_{k0} x_{k0} + \frac{1}{1+i} \sum_{k=0}^{n} p_{k1} x_{k1} = Y_0 + \frac{Y_1}{1+i}$$

- Let $C_t = \sum_{k=0}^{n} p_{kt} x_{kt}$ be nominal consumption at time t.
- Budget constraint becomes

$$C_0 + \frac{C_1}{1+i} = Y_0 + \frac{Y_1}{1+i} \equiv W$$

where W is nominal wealth.

Constraints in real terms

- Let $P_t = (\alpha_0^\sigma p_{0t}^{1-\sigma} + \cdots + \alpha_n^\sigma p_{nt}^{1-\sigma})^{\frac{1}{1-\sigma}}$ be the price index at time t
- Notice that $\frac{P_1}{P_0(1+i)} = \frac{1+\pi}{1+i} = \frac{1}{1+r}$, where π is the inflation rate, and r the real interest rate.
- Divide budget constraint by price index P_0

$$\frac{C_0}{P_0} + \frac{P_1}{P_0(1+i)}\frac{C_1}{P_1} = \frac{Y_0}{P_0} + \frac{P_1}{P_0(1+i)}\frac{Y_1}{P_1} = \frac{W}{P_0}$$
$$c_0 + \frac{c_1}{1+r} = y_0 + \frac{y_1}{1+r} = w$$

  where c_t is real consumption, y_t is real income, and w is real wealth.
- Constraint says that present value of real (composite) consumption equals the present value of real income.

Solving the problem: 2 steps

- Let $\tilde{U}$ denote CES function: $\tilde{U}(\mathbf{x}_t) = (\alpha_0 x_{0t}^\rho + \cdots + \alpha_n x_{nt}^\rho)^{\frac{1}{\rho}}$
- Utility becomes:

$$U = \frac{\tilde{U}(\mathbf{x}_0)^{1-\gamma}}{1-\gamma} + \beta \frac{\tilde{U}(\mathbf{x}_1)^{1-\gamma}}{1-\gamma}$$

- Consumer has to choose $2n + 2$ variables, subject to 1 budget constraint.
- To solve this problem, consumer makes decisions in two stages
    - Intra-temporal stage: Given all prices and the total amount to spend in each period, consumer chooses goods for each period separately.
    - Inter-temporal stage: Taking the intra-temporal solution as given, solve the inter-temporal problem:

Intra-temporal stage

- Given all prices and the total amount to spend in each period, consumer chooses goods for each period separately.
- Since intra-temporal preferences are CES, we know that if consumer spends C_t dollars and price level is P_t, the optimal utility he can get is

$$\tilde{V}(C_t, P_t) \equiv \max_{\mathbf{x_t}} \tilde{U}(\mathbf{x}_t) = \frac{C_t}{P_t} = c_t$$

Inter-temporal stage

- Taking the intra-temporal solution as given, problem becomes:

$$\max_{c_0, c_1} \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \frac{u_1^{1-\gamma}}{1-\gamma} \qquad \text{s.t} \qquad c_0 + \frac{c_1}{R} = w$$

- But this is equivalent to what we solved in previous section. Its solution is characterized by the Euler equation

$$c_0^{-\gamma} = \beta R c_1^{-\gamma} \quad \Rightarrow c_1 = (\beta R)^{1/\gamma} c_0$$

- Solution is

$$c_0 = \frac{R}{R + (\beta R)^{1/\gamma}} w \qquad c_1 = \frac{R(\beta R)^{1/\gamma}}{R + (\beta R)^{1/\gamma}} w$$

Marshallian demands for the goods

- Demands for each of the goods in then:

$$x_{kt} = \left(\frac{\alpha_k}{\frac{p_{kt}}{P_t}}\right) c_t$$
$$= \left(\frac{\alpha_k}{\frac{p_{kt}}{P_t}}\right) \frac{R(\beta R)^{t/\gamma}}{R + (\beta R)^{1/\gamma}} w$$

- Notice that demand for goods depends only on preference parameters (α_k) and real variables (wealth w, interest rates r, relative prices p_{kt}/P_t)

Modeling implications

- If utility is time-separable, we can split the problem of choosing n goods over T periods into $n+1$ problems:
  - decide how much to spend in each of the T periods (inter-temporal allocation); and
  - take each period budget and decide how to spend it into the n goods (intra-temporal allocation)
- If intra-temporal preference is CES, we can interpret the indirect utilities of the intra-temporal allocations as real composite consumption good.
- From now on, in our macro models we will analyze dynamic consumption behavior assuming that there exist such real composite consumption good.
- We will simply call it the consumption good.

Intertemporal consumption with uncertainty

Intertemporal consumption with uncertainty

- Representative consumer lives 2 periods.
- She can save and borrow at interest rate r.
- Her initial asset is a_0.
- She doesn't leave any debt or inheritance ($a_2 = 0$).
- Her income $y_t \geq 0$ in period $t = 0, 1$:
  - y_0 is known at time of deciding c_0.
  - $\tilde{y}_1$ is uncertain. It takes value y_{1s} with probability π_s, depending on the state of nature $s = 1, \ldots, S$.
  - Notice that $\sum_{s=1}^{S} \pi_s = 1$.
- Her expected future income is then

$$\mathbb{E}\,\tilde{y}_1 = \sum_{s=1}^{S} \pi_s y_{1s}$$

Budget constraint

- Budget constraints:

$$a_1 = R(a_0 + y_0 - c_0)$$
$$a_2 = R(a_1 + \tilde{y}_1 - \tilde{c}_1) = 0$$

- a_0 and y_0 are certain (she already have them in her bank).
- c_0 and a_1 are certain (she nows what she is choosing now).
- c_1 is uncertain because she needs to adjust future consumption to income shocks:

$$\tilde{c}_1 = a_1 + \tilde{y}_1 \quad \Rightarrow$$
$$\mathbb{E}\,\tilde{c}_1 = a_1 + \mathbb{E}\,\tilde{y}_1 \quad \Rightarrow$$
$$\tilde{c}_1 = \mathbb{E}\,\tilde{c}_1 + \underbrace{\tilde{y}_1 - \mathbb{E}\,\tilde{y}_1}_{\text{forecast error}}$$

Consumption plans, contingent on income

State	$\mathbb{P}$	Period 0	Period 1
s	π_s	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_{1s} = a_1 + y_{1s}$

Example 5:

Only two states of nature

State	Probability	Period 0	Period 1
L	π_L	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_1^L = a_1 + y_1^L$
H	π_H	$c_0 = a_0 + y_0 - \frac{a_1}{R}$	$c_1^H = a_1 + y_1^H$

Consumer wants to maximize her discounted expected utility:

$$
\begin{aligned}
U(c_0, c_1^L, c_1^H, \pi_L, \pi_H) &= \mathbb{E}_{\tilde{y}_2}\left[u(c_0) + \beta u(c_1)\right] \\
&= \pi_L\left[u(c_0) + \beta u(c_1^L)\right] + \pi_H\left[u(c_0) + \beta u(c_1^H)\right] \\
&= (\pi_L + \pi_H)u(c_0) + \beta\left[\pi_L u(c_1^L) + \pi_H u(c_1^H)\right] \\
&= u(c_0) + \beta\,\mathbb{E}\,u(c_1)
\end{aligned}
$$

$$
\begin{aligned}
U &= \{u(c_0) + \beta\, \mathbb{E}\, u(c_1)\} \\
&= \left\{u(c_0) + \beta \left[\pi_L u(c_1^L) + \pi_H u(c_1^H)\right]\right\} \\
&= \left\{u\left(a_0 + y_0 - \tfrac{a_1}{R}\right) + \beta \left[\pi_L u\left(a_1 + y_1^L\right) + \pi_H u\left(a_1 + y_1^H\right)\right]\right\}
\end{aligned}
$$

Objective now depends on a_1 alone. Take FOC:

$$
\begin{aligned}
0 &= -\tfrac{1}{R} u'\left(c_0\right) + \beta \pi_L u'(c_1^L) + \beta \pi_H u'(c_1^H) \\
u'\left(c_0\right) &= \beta R \left[\pi_L u'(c_1^L) + \pi_H u'(c_1^H)\right] \\
&= \beta R\, \mathbb{E}\left[u'(c_1)\right] \qquad \text{(Euler equation)}
\end{aligned}
$$

Wealth and permanent income

- Combining the budget constraints she gets

$$c_0 + \frac{\tilde{c}_1}{R} = \underbrace{a_0 + y_0 + \frac{\tilde{y}_1}{R}}_{\text{wealth } \tilde{W}_0} \quad \text{(for any possible state of nature)}$$

- Her wealth at time 0 is uncertain because future income is random. But she can form an expectation:

$$c_0 + \frac{\mathbb{E}\,\tilde{c}_1}{R} = a_0 + y_0 + \frac{\mathbb{E}\,\tilde{y}_1}{R} = \mathbb{E}\,\tilde{W}_0$$

- Her permanent income y_p is the constant level of consumption that she expects to be able to afford, given her expected wealth. Then

$$y_p = \frac{R}{1+R}\,\mathbb{E}\,\tilde{W}$$

The consumer's problem

- She wants to maximize her discounted expected utility (von Neumann-Morgenstern):

$$
\begin{aligned}
U\left(c_0, \{c_{1s}; \pi_s\}_{s=1}^{S}\right) &= \mathbb{E}_{\tilde{y}_2}\left[u(c_0) + \beta u(c_1)\right] \\
&= u(c_0) + \beta\, \mathbb{E}\, u(c_1)
\end{aligned}
$$

- subject to contingent plans

$$
c_0 + \frac{c_{1s}}{R} = a_0 + y_0 + \frac{y_{1s}}{R} \equiv W_s \qquad \text{(for } s = 1, \ldots, S\text{)}
$$

- There are S constraints (one per state of nature).
- Let $\lambda_s \pi_s$ be the Lagrange multiplier associated with the s^{th} constraint.

Solving the problem

- The Lagrangian is

$$\begin{aligned}\mathcal{L} &= u(c_0) + \beta\,\mathbb{E}\,u(c_1) + \sum\nolimits_s \lambda_s \pi_s \left(W_s - c_s - \frac{c_{1s}}{R}\right) \\ &= u(c_0) + \sum\nolimits_s \pi_s \left[\beta u(c_{1s}) + \lambda_s \left(W_s - c_s - \frac{c_{1s}}{R}\right)\right]\end{aligned}$$

- FOCs:

$$\begin{aligned}&(\text{wrt } c_0) \qquad 0 = u'(c_0) - \sum\nolimits_s \pi_s \lambda_s \quad \Rightarrow \quad u'(c_0) = \mathbb{E}\,\lambda \\ &(\text{wrt } c_{1s}) \qquad 0 = \pi_s \left[\beta u'(c_{1s}) - \frac{\lambda_s}{R}\right] \quad \Rightarrow \quad \pi_s \beta R u'(c_{1s}) = \pi_s \lambda_s\end{aligned}$$

The Euler equation

- Adding up the FOCs wrt c_{1s}, we get

$$\sum_s \pi_s \beta R u'(c_{1s}) = \sum_s \pi_s \lambda_s$$
$$\beta R \, \mathbb{E} \, u'(c_1) = \mathbb{E} \, \lambda$$

- Substituting $\mathbb{E} \, \lambda$ from the first FOC to get

Euler equation	$u'(c_0) = \beta R \, \mathbb{E} \, u'(c_1)$

Side note: Some math worth remembering

- Let u and v be functions, X and Z random variables, and a and b scalars.
- Suppose that X and Z depend on parameter t.
- Then, under fairly general conditions:

$$\mathbb{E}\left[au(X) + bv(Z)\right] = a\,\mathbb{E}\,u(X) + b\,\mathbb{E}\,v(Z)$$

$$\frac{\partial\,\mathbb{E}\,u(X)}{\partial t} = \mathbb{E}\left[u'(X)\frac{\partial X}{\partial t}\right]$$

A faster way to get the Euler equation

- Instead of having one constraint for each state of nature, just write one: the expected values of the constraint:

$$c_0 + \frac{\mathbb{E}\,\tilde{c}_1}{R} = \mathbb{E}\,\tilde{W}_0$$

- Just keep in mind that this is a shortcut: the budget constraint must be satisfied in every state of nature, not only in expected values.
- Besides, the consumer is choosing future consumption contingent on each state of nature. She is not just choosing her expected future consumption.

Solving the problem

- Lagrangian is

$$\mathcal{L} = u'(c_0) + \beta\,\mathbb{E}\,u(c_1) + \lambda\left(\mathbb{E}\,\tilde{W} - c_0 - \frac{\mathbb{E}\,c_1}{R}\right)$$

- FOCs

$$\begin{aligned} &(\text{wrt } c_0) \qquad 0 = u'(c_0) - \lambda \qquad && \Rightarrow \quad u'(c_0) = \lambda \\ &(\text{wrt } c_1) \qquad 0 = \beta\,\mathbb{E}\,u'(c_1) - \frac{\lambda}{R} \qquad && \Rightarrow \quad \beta R\,\mathbb{E}\,u'\left(c_1\right) = \lambda \end{aligned}$$

Euler equation, again

- Then, from the two FOCs

$$u'(c_0) = \beta R \, \mathbb{E} \, u'(c_1) \qquad \text{(Euler equation)}$$

- Euler equation can be written as:

$$\underbrace{\frac{u'(c_0)}{\beta \, \mathbb{E} \, u'(c_1)}}_{\text{MRS of present consumption for future consumption}} = \underbrace{R}_{\text{price of present consumption in terms of future consumption}}$$

Example 6:

Hall 1978

- Assume that utility is quadratic $u(c) = \alpha c - 0.5c^2$ and that $\beta R = 1$.
- Euler equation is:

$$\mathbb{E}\, c_1 = c_0$$

- This means that consumption would follow a random walk.
- In such case, under the pure life cycle-permanent income hypothesis, a forecast of future consumption obtained by extrapolating today's level by the historical trend is impossible to improve.

Example 7:

CRRA utility, with uncertainty

- Now assume that consumer has constant relative risk aversion: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 0$.
- Euler equation is:

$$c_0^{-\sigma} = \beta R \, \mathbb{E} \left(c_1^{-\sigma} \right)$$

- But notice that $\mathbb{E} \left(c_1^{-\sigma} \right) \neq (\mathbb{E} \, c_1)^{-\sigma}$, so we can not simply use budget constraint

$$c_0 + \frac{\mathbb{E} \, \tilde{c}_1}{R} = \mathbb{E} \, \tilde{W}_0$$

  to solve for c_0 and $\mathbb{E} \, c_1$.
- So, in dynamic models with uncertainty, it is often necessary to use numerical methods to analyze the solution of the model.

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