

# Lecture 11

Markov processes

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## Introduction

- Markov processes are an indispensable ingredient of DSGE models.
- They preserve the recursive structure that these models inherit from their deterministic relatives.
- In this lecture we review a few results about these processes that we will need repeatedly in the modeling of business cycles.

## 1 Stochastic process

### Stochastic Process

A *stochastic process* is a time sequence of random variables  $\{Y_t\}_{t=0}^{\infty}$ .

Two types of processes:

**Continuous** if realizations are taken from an interval of the real line  $Y_t \in [a, b] \subseteq \mathbb{R}$ .

**Discrete** if there is a countable number of realizations  $Y_t \in \{y_1, y_2, \dots, y_n\}$ .

### i.i.d. Stochastic Process

- The elements of a stochastic process are *identically and independently distributed (iid for short)*, if the probability distribution is the same for each member of the process  $Z_t$  and independent of the realizations of other members of the process.
- In this case

$$\mathbb{P}[Y_1 = y_1, Y_2 = y_2, \dots, Y_T = y_T] = \mathbb{P}(Y_1 = y_1) \times \mathbb{P}(Y_2 = y_2) \times \dots \times \mathbb{P}(Y_T = y_T)$$

### Unconditional moments

- Unconditional cumulative distribution function

$$F_{Y_t}(y) = \mathbb{P}[Y_t \leq y]$$

- Unconditional expectation (mean)

$$\mu_t \equiv \mathbb{E}(Y_t) = \int_{-\infty}^{\infty} y \, dF_{Y_t}(y)$$

- Unconditional variance

$$\gamma_{0t} \equiv \mathbb{E}(Y_t - \mu_t)^2 = \int_{-\infty}^{\infty} (y - \mu_t)^2 \, dF_{Y_t}(y)$$

- Autocovariance

$$\gamma_{jt} \equiv \mathbb{E}(Y_t - \mu_t)(Y_{t-j} - \mu_{t-j})$$

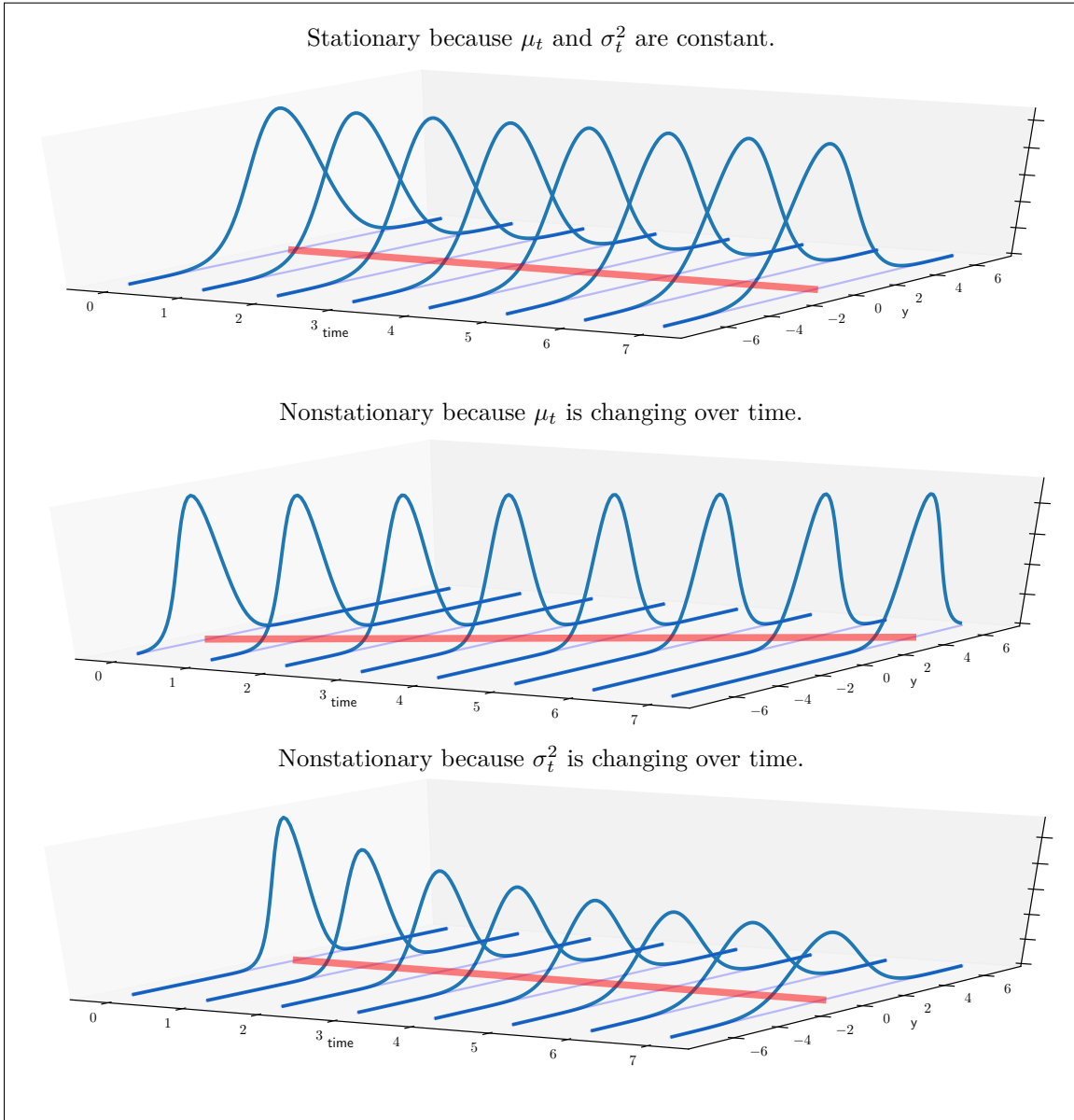
### Stationarity

If neither the mean  $\mu_t$  nor the autocovariances  $\gamma_{jt}$  depend on the date  $t$ , then the process for  $Z_t$  is said to be *covariance-stationary* or *weakly stationary*:

$$\begin{aligned} \mathbb{E}(Y_t) &= \mu && \text{for all } t \\ \mathbb{E}(Y_t - \mu)(Y_{t-j} - \mu) &= \gamma_j && \text{for all } t \text{ and any } j \end{aligned}$$

### Example 1: Stationary and nonstationary processes

Suppose  $Y_t$  is a stochastic process such that  $Y_t \sim N(\mu_t, \sigma_t^2)$



### White noise

- The basic building block for the processes considered in this lecture is a sequence  $\{\epsilon_t\}$  whose elements have mean zero and variance  $\sigma^2$ ,

$$\mathbb{E}(\epsilon_t) = 0 \quad (\text{zero mean})$$

$$\mathbb{E}(\epsilon_t^2) = \sigma^2 \quad (\text{constant variance})$$

$$\mathbb{E}(\epsilon_t \epsilon_\tau) = 0 \quad \text{for } t \neq \tau \quad (\text{uncorrelated terms})$$

- If the terms are normally distributed

$$\epsilon_t \sim N(0, \sigma^2)$$

then we have the *Gaussian white noise process*.

## 2 The first-order autoregressive process

### Definition of a AR(1) process

- A first-order autoregression, denoted AR(1), satisfies the following difference equation:

$$Y_t = c + \phi Y_{t-1} + \epsilon_t$$

where  $\{\epsilon_t\}$  is a white noise sequence.

- It is stationary if and only if  $|\phi| < 1$ .
- In what follows, we assume the process is stationary.

### MA( $\infty$ ) representation of a AR(1) process

- If the AR(1) process is stationary, it can be written

$$Y_t = \frac{c}{1 - \phi} + \epsilon_t + \phi \epsilon_{t-1} + \phi^2 \epsilon_{t-2} + \phi^3 \epsilon_{t-3} + \dots$$

### Conditional versus unconditional mean

- The *conditional* mean given the previous observation is

$$\mathbb{E}[Y_t | Y_{t-1}] = c + \phi Y_{t-1}$$

- The *unconditional* mean is

$$\mu \equiv \mathbb{E}[Y_t] = \frac{c}{1 - \phi}$$

- Since  $c = (1 - \phi)\mu$ , the AR(1) process can be written as *deviations from 'equilibrium'*

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \epsilon_t$$

### Impulse-response

- Starting with  $Y_{t-1}$ , the value of  $Y_{t+s}$  will be

$$Y_{t+s} - \mu = \phi^{s+1}(Y_{t-1} - \mu) + \phi^s \epsilon_t + \phi^{s-1} \epsilon_{t+1} + \dots + \phi \epsilon_{t+s-1} + \epsilon_{t+s}$$

- Suppose that starting in 'equilibrium' ( $Y_{t-1} - \mu = 0$ ) there is a time- $t$  *transitory* shock ( $\epsilon_t = \nu$ ) but no more shocks thereafter ( $\epsilon_{t+1} = \dots = \epsilon_{t+s} = 0$ ). Then

$$Y_{t+s} - \mu = \phi^s \nu$$

- This is known as an *impulse-response function*.
- Notice that the process will return to equilibrium as long as  $|\phi| < 1$ .

### Conditional versus unconditional variance

- The *conditional* variance given the previous observation is

$$\text{Var}[Y_t | Y_{t-1}] = \text{Var}[c + \phi Y_{t-1} + \epsilon_t | Y_{t-1}] = \sigma^2$$

- The *unconditional* mean is

$$\gamma_0 \equiv \text{Var}[Y_t] = \frac{\sigma^2}{1 - \phi^2}$$

- Notice that  $\gamma_0 > \text{Var}[Y_t | Y_{t-1}]$

### Autocovariance and autocorrelation

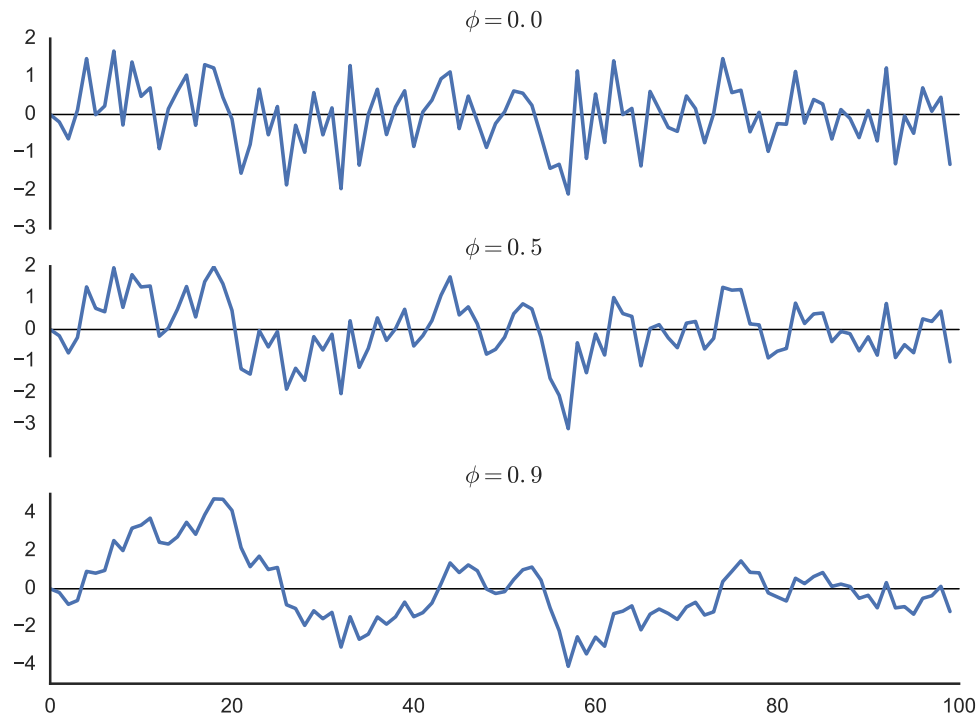
- The *autocovariance* is given by

$$\gamma_j = \phi^j \gamma_0 \quad (j = 1, 2, \dots)$$

- The *autocorrelation* is given by

$$\rho_j = \phi^j \quad (j = 1, 2, \dots)$$

#### Example 2: Realizations of an AR(1) process



The three processes are built from the same white noise realization. Notice how process becomes more persistent as  $\phi$  approaches 1.

## 3 Markov chains

### Markov property

A stochastic process  $\{Z_t\}_{t=0}^{\infty}$  has the *Markov property* if for all  $k \geq 1$  and all  $t$ ,

$$\mathbb{P}[Z_{t+1} | Z_t, Z_{t-1}, \dots, Z_{t-k}] = \mathbb{P}[Z_{t+1} | Z_t]$$

That is, the the probability distribution of  $Z_{t+1}$  only depends upon the realization of  $Z_t$ .

#### Example 3: AR(1) process

- The  $AR(1)$  process is a Markov process:

$$Z_t = (1 - \rho)\bar{Z} + \rho Z_{t-1} + \epsilon_t$$

where  $\rho \in [0, 1)$ , and  $\epsilon_t \sim \text{iid}N(0, \sigma^2)$  is a *white noise* process.

- Given  $Z_t$ , next period's variable  $Z_{t+1}$  is normally distributed with:

**mean:**  $\mathbb{E}(Z_{t+1} | Z_t) = (1 - \rho)\bar{Z} + \rho Z_t$

**variance:**  $\text{Var}(Z_{t+1} | Z_t) = \sigma^2$

### Markov Chains

*Markov chains* are discrete valued Markov processes. They are characterized by three objects:

- The  $n$  different *realizations* of  $Z_t$ , represented by the column vector  $z = [z_1, z_2, \dots, z_n]'$ .
- The *probability distribution of the initial date*  $t = 0$ ,  $\pi_0 = [\pi_{01}, \pi_{02}, \dots, \pi_{0n}]'$ , where  $\pi_{0i} = \mathbb{P}[Z_0 = z_i]$ .
- The *transition matrix*  $P = (p_{ij})$ , where  $p_{ij} = \mathbb{P}[Z_{t+1} = z_j | Z_t = z_i]$ , representing the dynamics of the process.

Notice that

- $p_{ij} \geq 0$  and  $\sum_{j=1}^n p_{ij} = 1$ .
- $\pi_{0i} \geq 0$  and  $\sum_{i=1}^n \pi_{0i} = 1$ .

### Example 4: Unemployment

A worker can either be employed or unemployed:

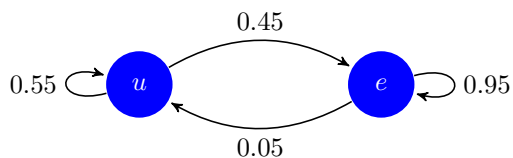
- If unemployed, she will get a job with probability  $p = 45\%$
- If employed, she will lose her job with probability  $q = 5\%$

The worker is employed at  $t = 0$ . Then the Markov chain is:

**outcomes** {unemployed, employed} or  $z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

**initial probability**  $\pi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

**transition probability**  $P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \\ 0.05 & 0.95 \end{bmatrix}$



## Example 5: Credit ratings

Transition of the credit ratings from one year to the next:

	AAA	AA	A	BBB	BB	B	CCC	D	N.R.
AAA	90.34	5.62	0.39	0.08	0.03	0	0	0	3.5
AA	0.64	88.78	6.72	0.47	0.06	0.09	0.02	0.01	3.21
A	0.07	2.16	87.94	4.97	0.47	0.19	0.01	0.04	4.16
BBB	0.03	0.24	4.56	84.26	4.19	0.76	0.15	0.22	5.59
BB	0.03	0.06	0.4	6.09	76.09	6.82	0.96	0.98	8.58
B	0	0.09	0.29	0.41	5.11	74.62	3.43	5.3	10.76
CCC	0.13	0	0.26	0.77	1.66	8.93	53.19	21.94	13.14
D	0	0	0	0	1	3.1	9.29	51.29	37.32
N.R.	0	0	0	0	0	0.1	8.55	74.06	17.07

Transition probabilities are expressed in %.

- Higher ratings are more stable: the diagonal coefficients of the matrix go decreasing.
- Starting from the rating AA it is easier to be downgraded (probability 6.72%) than to be upgraded (probability 0.64%).

### Transition over multiple periods

- The transition matrix is also called a *stochastic matrix*.
- It defines the probabilities of moving from one value of the state to another in one period.
- The probability of moving from one value of the state to another in two periods is determined by  $P^2$  because

$$\begin{aligned}
 \mathbb{P}[Z_{t+2} = z_j | Z_t = z_i] &= \sum_{h=1}^n \mathbb{P}[Z_{t+2} = z_j | Z_{t+1} = z_h] \times \mathbb{P}[Z_{t+1} = z_h | Z_t = z_i] \\
 &= \sum_{h=1}^n P_{ih} P_{hj} = P_{ij}^{(2)}
 \end{aligned}$$

### The unconditional distribution

The probability distribution of  $Z_t$  evolves according to  $\pi'_{t+1} = \pi'_t P$ . Therefore

$$\begin{aligned}
 \pi'_2 &= \pi'_0 P^2 \\
 &\vdots \\
 \pi'_k &= \pi'_0 P^k
 \end{aligned}$$

The limit for  $k \rightarrow \infty$  is the *time invariant, stationary, or ergodic* distribution of the Markov chain. It is defined by

$$\pi' = \pi' P \Leftrightarrow (I - P)\pi = 0$$

The limit exist and is independent of the initial distribution  $\pi_0$  if  $p_{ij}^{(k)} > 0$  for some integer  $k \geq 1$ .

### Example 6: Unemployment (cont.)

For the worker who can either be employed or unemployed according to Markov matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \\ 0.05 & 0.95 \end{bmatrix}$$

the stationary distribution  $[x \ 1-x]'$  is the solution to:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix} \right\} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} p & -q \\ -p & q \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then  $x = \frac{q}{p+q}$  and the stationary distribution is:  $\begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}$ . This means that the long run probability of being unemployed is 10%.

## References

- Hamilton, James M. (1994). *Time Series Analysis*. Princeton University Press. ISBN: 0-691-04289-6.  
Heer, Burkhard and Alfred Maubner (2009). *Dynamic General Equilibrium Modeling. Computational Methods and Applications*. 2nd ed. Springer-Verlag Berlin Heidelberg. 702 pp. ISBN: 978-3-642-03148-9.