# Lecture 15

## Dynamic Stochastic General Equilibrium Model

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# Introduction

## Dynamic Stochastic General Equilibrium (DSGE) models

- DSGE models have become the fundamental tool in current macroeconomic analysis
- They are in common use in academia and in central banks.
- Useful to analyze how economic agents respond to changes in their environment, in a dynamic general equilibrium micro-founded theoretical setting in which all endogenous variables are determined simultaneously.
- Static models and partial equilibrium models have limited value to study how the economy responds to a particular shock.

# DSGE: microfoundations + rational expectations

- Modern macro analysis is increasingly concerned with the construction, calibration and/or estimation, and simulation of DSGE models.
- DSGE models start from micro-foundations, taking special consideration of the rational expectation forward-looking economic behavior of agents.

# Households

## General assumptions about consumers

- There is a representative agent.
- Who is an optimizer: she maximizes a given objective function.
- She lives forever: infinite horizon
- Her happiness depends on consumption $C$ and leisure $O$.
- The maximization of her objective function is subject to a resource restriction: the budget constraint.

# Instant utility

- The instant utility function is

$$u(C, O)$$

- She prefers more consumption and more leisure to less:

$$u_C > 0 \qquad\qquad u_O > 0$$

- Higher consumption (and leisure) implies greater utility but at a decreasing rate:

$$u_{CC} < 0 \qquad\qquad u_{OO} < 0$$

# Expected utility function

- The consumer's happiness depends on the entire path of consumption and leisure that she expects to enjoy:

$$U(C_0, C_1, \ldots, C_\infty, O_0, O_1, \ldots, O_\infty)$$

- She's impatient: she discounts future utility by $\beta$.
- Her utility is time separable.
- Therefore, her expected utility is

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, O_t)$$

# Resource ownership

- To define a budget constraint we must introduce property rights.
- Here,we assume that the consumer is the owner of production factors: capital $K$ and $L$ labor.
- $L$ comes from the available endowment of time, which we normalize to 1. Because time cannot be accumulated, labor decisions will be static.
- $K$ is accumulated through investment, which in turn depends on savings.
- Consumer also owns the firm.

# The budget constraint

- Household income comes from renting both productive factors to the production sector, at given rental prices.
- Household can do two things with these earnings: expend it in consumption or save it.
- Then, the budget constraint is

$$P_t\left(C_t + S_t\right) \leq W_t L_t + R_t K_t + \Pi_t$$

  where

$$\begin{aligned} P_t &= \text{price of consumption good} & S_t &= \text{savings} \\ R_t &= \text{user cost of capital} & W_t &= \text{wage} \\ \Pi_t &= \text{firm's profits (= dividends)} \end{aligned}$$

- Since there is no money, we normalize $P_t = 1 \quad \forall t$.

# Resource constraints

- Since time is spent either working or in leisure:

$$O_t + L_t = 1 \qquad \forall t$$

- Given this constraint, in what follows we write the instant utility function as:

$$u(C, 1 - L)$$

- Because capital deteriorates over time, its accumulation is subject to depreciation rate $\delta$:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

# The financial sector

- To keep things simple, we assume that there is a competitive sector that transforms savings directly into investment without any cost.
- Thus

$$S_t = I_t$$

- Combining this assumption with the budget constraint and the capital accumulation equation, the consumer is constraint by

$$\begin{aligned} C_t &\leq W_t L_t + R_t K_t + \Pi_t - S_t \\ &\leq W_t L_t + R_t K_t + \Pi_t - I_t \\ &\leq W_t L_t + R_t K_t + \Pi_t + (1-\delta) K_t - K_{t+1} \\ &\leq W_t L_t + \Pi_t + (1 + R_t - \delta) K_t - K_{t+1} \end{aligned}$$

# The consumer problem

The consumer problem is to maximize her lifetime utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, 1 - L_t)$$

subject to the budget constraint*

$$C_t = W_t L_t + \Pi_t + (1 + R_t - \delta) K_t - K_{t+1} \qquad \forall t = 0, 1, \dots$$

where $K_0$ is predetermined.

*We impose equality because $u_C > 0$.

# The consumer problem: dynamic programming

- The consumer problem is recursive, so we can represent it by a Bellman equation.
- ”Current capital” is the state variable, ”next capital” and labor are the policy variables.
- Then we write

$$V(K) = \max_{K',L} \left\{ u(C, 1-L) + \beta\, \mathbb{E}\, V(K') \right\}$$

subject to the budget constraint

$$C = WL + \Pi + (1 + R - \delta)K - K'$$

# The consumer problem: solution

- The FOCs are:

$$u_O = W u_C \qquad \text{(wrt labor)}$$
$$u_C = \beta\,\mathbb{E}\,V'(K') \qquad \text{(wrt capital)}$$

- The envelope condition is

$$V'(K) = (1 + R - \delta)\,u_C$$

- Therefore, the Euler equation is

$$u_C = \beta\,\mathbb{E}\left[(1 + R' - \delta)u_{C'}\right]$$

## Consumer optimization: In summary

- For the numerical solution of the model, we assume that

$$u\left(C_t, 1-L_t\right) = \gamma \ln C_t + (1-\gamma)\ln(1-L_t)$$

- Therefore, the solution of the consumer problem requires

$$\begin{aligned}
1 &= \beta\,\mathbb{E}\left[(1+R_{t+1}-\delta)\frac{C_t}{C_{t+1}}\right] \\
C_t &= \frac{\gamma}{1-\gamma} W(1-L_t) \\
K_{t+1} &= (1-\delta)K_t + I_t
\end{aligned}$$

# Firms

# The firms

- Firms produce goods and services the households will consume of save.
- To do this, they transform capital $K$ and labor $L$ into final output.
- They rent these factors from households.

# Production function

- Technology is described by the aggregate production function

$$Y_t = A_t F\left(K_t, L_t\right)$$

where $Y_t$ is aggregate output and $A_t$ is total factor productivity (TFP).

- Production increases with inputs...

$$F_K > 0 \qquad\qquad F_L > 0$$

- ...but marginal productivity of each factor is decreasing :

$$F_{KK} < 0 \qquad\qquad F_{LL} < 0$$

# Production function (cont'n)

We assume that

- Production has constant returns to scale:

$$A_t F\left(\lambda K_t, \lambda L_t\right) = \lambda Y_t$$

- Both factors are indispensable for production

$$A_t F\left(0, L_t\right) = 0 \qquad\qquad A_t F\left(K_t, 0\right) = 0$$

- Production satisfies the Inada conditions

$$\lim_{K \to 0} F_K = \infty \qquad\qquad \lim_{L \to 0} F_L = \infty$$

$$\lim_{K \to \infty} F_K = 0 \qquad\qquad \lim_{L \to \infty} F_L = 0$$

# The firm's problem: static optimization

- Firms maximize profits, subject to the technological constraint.

$$\max_{K_t,L_t} \Pi_t = Y_t - W_t L_t - R_t K_t$$
$$\text{s.t.} \quad Y_t = A_t F\left(K_t, L_t\right)$$

or simply

$$\max_{K_t,L_t} A_t F\left(K_t, L_t\right) - W_t L_t - R_t K_t$$

## The firm's problem: solution

- The FOCs are:

$$W_t = A_t F_L (K_t, L_t) \qquad \text{(wrt labor)}$$
$$R_t = A_t F_K (K_t, L_t) \qquad \text{(wrt capital)}$$

  that is, the relative price of productive factors equals their marginal productivity.

# Side note: Euler's theorem

Let $f(x)$ be a $\mathcal{C}^1$ homogeneous function of degree $k$ on $\mathbb{R}^n_+$. Then, for all $x$,

$$x_1 \frac{\partial f}{\partial x_1}(x) + x_2 \frac{\partial f}{\partial x_2}(x) + \cdots + x_n \frac{\partial f}{\partial x_n}(x) = kf(x)$$

## The firm's profits

- Since $F$ is homogeneous of degree one (constant returns to scale), Euler's theorem implies

$$[A_t F_K (K_t, L_t)]\, K_t + [A_t F_L (K_t, L_t)]\, L_t = Y_t$$

- Substitute FOCs from firms problem:

$$R_t K_t + W_t L_t = Y_t$$

- and therefore optimal profits will equal zero:

$$\Pi_t = Y_t - R_t K_t - W_t L_t = 0$$

# The total factor productivity

- The TFP $A_t$ follows a first-order autorregresive process:

$$\ln A_t = (1-\rho)\ln \bar{A} + \rho \ln A_{t-1} + \epsilon_t$$

  where the productivity shock $\epsilon_t$ is a Gaussian white noise process:

$$\epsilon_t \sim N(0, \sigma^2)$$

- This assumption led to the birth of the Real Business Cycle (RBC) literature.

- The TFP process can also be written

$$\ln A_t - \ln \bar{A} = \rho \left(\ln A_{t-1} - \ln \bar{A}\right) + \epsilon_t$$

- In equilibrium, $A_t = \bar{A}$.
- Productivity shocks cause persistent deviations in productivity from its equilibrium value:

$$\frac{\partial \left(\ln A_{t+s} - \ln \bar{A}\right)}{\partial \epsilon_t} = \rho^{s-1} > 0$$

  as long as $\rho > 0$.
- Although persistent, the effect of a shock is not permanent

$$\lim_{s \to \infty} \frac{\partial \left(\ln A_{t+s} - \ln \bar{A}\right)}{\partial \epsilon_t} = \rho^{s-1} = 0$$

# Firm optimization: In summary

- For the numerical solution of the model, we assume that

$$A_t F\left(K_t, L_t\right) = A_t K_t^{\alpha} L_t^{1-\alpha} \qquad \text{and} \qquad \bar{A} = 1$$

- Therefore, the solution of the firm problem requires

$$\begin{aligned}
W_t &= (1-\alpha) A_t \left(\frac{K_t}{L_t}\right)^{\alpha} = (1-\alpha)\frac{Y_t}{L_t} \\
R_t &= \alpha A_t \left(\frac{L_t}{K_t}\right)^{1-\alpha} = \alpha \frac{Y_t}{K_t} \\
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
\ln A_t &= \rho \ln A_{t-1} + \epsilon_t
\end{aligned}$$

# The competitive equilibrium

# Putting the agents together

- The equilibrium of this models depends on the interaction of consumers and firms.
- Households decide how much to consume $C_t$, to invest (save) $I_t = S_t$, and to work $L_t$, with the objective of maximizing their happiness, taking as given the prices of inputs.
- Firms decide how much to produce $Y_t$, by hiring capital $K_t$ and labor $L_t$, given the prices of production factors.
- Since both agents take all prices as given, this is a competitive equilibrium.

# The competitive equilibrium

The competitive equilibrium for this economy consists of

1. A pricing system for $W$ and $R$
2. A set of values assigned to $Y$, $C$, $I$, $L$ and $K$.

such that

1. given prices, the consumer optimization problem is satisfied;
2. given prices, the firm maximizes its profits; and
3. all markets clear at those prices.

The competitive equilibrium for this economy consists of prices $W_t$ and $R_t$, and quantities $A_t$, $Y_t$, $C_t$, $I_t$, $L_t$ and $K_{t+1}$ such that:

$$
\begin{aligned}
1 &= \beta\,\mathbb{E}\left[(1 + R_{t+1} - \delta)\frac{C_t}{C_{t+1}}\right] \\
C_t &= \frac{\gamma}{1-\gamma} W(1 - L_t) \\
K_{t+1} &= (1-\delta)K_t + I_t \\
W_t &= (1-\alpha)\frac{Y_t}{L_t} \\
R_t &= \alpha\frac{Y_t}{K_t} \\
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
\ln A_t &= \rho \ln A_{t-1} + \epsilon_t \\
Y_t &= C_t + I_t
\end{aligned}
$$

- First 3 equations characterize solution of consumer problem
- Next 3 equations characterize solution of firm problem
- Next equation governs dynamic of TFP
- Last equation implies equilibrium in goods markets
- Equilibrium in factor markets is implicit: we use same $K$, $L$ in consumer and firm problems
- Later, we use these 8 equations in IRIS to solve and simulate the model.

# Welfare theorems

If there are no distortions such as (distortionary) taxes or externalities, then

$1^{st}$ **Welfare Theorem** The competitive equilibrium characterized in last slide is Pareto optimal

$2^{nd}$ **Welfare Theorem** For any Pareto optimum a price system $W_t$, $R_t$ exists which makes it a competitive equilibrium

# The central planning equilibrium

# The central planner

- An alternative setting to a competitive market environment is to consider a centrally planned economy
- The central planner makes all decisions in the economy.
- Objective: the joint maximization of social welfare
- Prices have no role in this setting.

## The central planner problem

The central planner problem is to maximize social welfare

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, 1 - L_t)$$

subject to the constraints $\qquad \forall t = 0, 1, \ldots$

$$\begin{aligned} C_t + I_t &= Y_t && \text{(resource constraint)} \\ Y_t &= A_t F(K_t, L_t) && \text{(technology constraint)} \\ K_{t+1} &= (1-\delta)K_t + I_t && \text{(capital accumulation)} \end{aligned}$$

where $K_0$ is predetermined. The three constraints can be combined into

$$C_t + K_{t+1} = A_t F(K_t, L_t) + (1-\delta)K_t$$

# The central planner problem: dynamic programming

- The central planner problem is recursive too, so we can represent it by a Bellman equation.
- "Current capital" is the state variable, "next capital" and labor are the policy variables.
- Then we write

$$V(K) = \max_{K',L} \left\{ u(C, 1-L) + \beta \, \mathbb{E} \, V(K') \right\}$$

subject to the constraint

$$C = AF(K, L) + (1-\delta)K - K'$$

# The central planner problem: solution

- The FOCs are:

$$
\begin{aligned}
u_O &= u_C A F_L(K, L) && \text{(wrt labor)} \\
u_C &= \beta\, \mathbb{E}\, V'(K') && \text{(wrt capital)}
\end{aligned}
$$

- The envelope condition is

$$V'(K) = [A F_K(K, L) + 1 - \delta]\, u_C$$

- Therefore, the Euler equation is

$$u_C = \beta\, \mathbb{E} \left\{ \big[ A F_{K'}(K', L') + 1 - \delta \big]\, u_{C'} \right\}$$

# Central planner optimization: In summary

- For the numerical solution of the model, we assume again that

$$u\left(C_t, 1-L_t\right) = \gamma \ln C_t + (1-\gamma)\ln(1-L_t)$$
$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

- Therefore, the solution of the central planner problem requires

$$1 = \beta\, \mathbb{E}\left[\left(\alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right) \frac{C_t}{C_{t+1}}\right]$$
$$C_t = \frac{\gamma}{1-\gamma}(1-\alpha)\frac{Y_t}{L_t}(1-L_t)$$
$$K_{t+1} = (1-\delta)K_t + I_t$$

# The central planning equilibrium

The central planning equilibrium for this economy consists quantities $A_t$, $Y_t$, $C_t$, $I_t$, $L_t$ and $K_{t+1}$ such that:

$$
\begin{aligned}
1 &= \beta\,\mathbb{E}\left[\left(\alpha\frac{Y_{t+1}}{K_{t+1}}+1-\delta\right)\frac{C_t}{C_{t+1}}\right] \\
\frac{C_t}{1-L_t} &= \frac{\gamma}{1-\gamma}(1-\alpha)\frac{Y_t}{L_t} \\
K_{t+1} &= (1-\delta)K_t + I_t \\
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
\ln A_t &= \rho \ln A_{t-1} + \epsilon_t \\
Y_t &= C_t + I_t
\end{aligned}
$$

- These equations characterize solution of the social planner problem
- There are no market equilibrium conditions, because there are no markets
- There are no prices
- Last equation is a feasibility constraint

# Central planner vs. competitive market equilibria

- The solution under a centrally planned economy is exactly the same as under a competitive market.
- This is because there are no distortions in our model that alters the agents' decisions regarding the efficient outcome.
- Only difference: In central planner setting there are no markets for production factors, and therefore no price for factors either.

# The steady state

# The steady state

- The steady state refers to a situation in which, in the absence of random shocks, the variables are constant from period to period.
- Since there is no growth in our model, it is stationary, and therefore it has a steady state.
- We can think of the stead state as the long term equilibrium of the model.
- To calculate the steady state, we set all shocks to zero and drop time indices in all variables.

# Computing the steady state for the competitive equilibrium

In this case, the steady state consists of prices $\bar{W}$ and $\bar{R}$, and quantities $\bar{A}$, $\bar{Y}$, $\bar{C}$, $\bar{I}$, $\bar{L}$ and $\bar{K}$ such that:

$$1 = \beta(1 + \bar{R} - \delta) \qquad \bar{C} = \frac{\gamma}{1-\gamma}\bar{W}(1-\bar{L})$$

$$\bar{I} = \delta\bar{K} \qquad \bar{W} = (1-\alpha)\frac{\bar{Y}}{\bar{L}}$$

$$\bar{R} = \alpha\frac{\bar{Y}}{\bar{K}} \qquad \bar{Y} = \bar{A}\bar{K}^{\alpha}\bar{L}^{1-\alpha}$$

$$\bar{A} = 1 \qquad \bar{Y} = \bar{C} + \bar{I}$$

# IRIS

# IRIS

- IRIS is a free, open-source toolbox for macroeconomic modeling and forecasting in Matlab®, developed by the IRIS Solutions Team since 2001.
- In a user-friendly command-oriented environment, IRIS integrates core modeling functions (flexible model file language, tools for simulation, estimation, forecasting and model diagnostics) with supporting infrastructure (time series analysis, data management, or reporting).
- It can be downloaded from Github.

# Solving the model with IRIS

- To solve the model, one creates two files

  **.model** Here we describe the model: declare its variables, parameters, and equations

  **.m** this is a regular MATLAB file. Here we load the model, solve it, and analyze it.

- The code presented here is based on Torres (2015, pp.51-52), which was written to be used with DYNARE instead of IRIS.

# The .model file: Define variables

- The .model file is a text file, where we declare (usually) four sections:
  - `!transition_variables`
  - `!transition_shocks`
  - `!parameters`
  - `!transition_equations`
- Although not required, using `'labels'` greatly improves readability.

```
!transition_variables
'Income' Y
'Consumption' C
'Investment' I
'Capital' K
'Labour' L
'Wage' W
'Real interest rate' R
'Productivity' A

!transition_shocks
'Productivity shock' e
```

# Model calibration

For the model to be completely computationally operational, a value must be assigned to the parameters.

| Parameter | Definition | Value |
|---|---|---|
| $\alpha$ | Marginal product of capital | 0.35 |
| $\beta$ | Discount factor | 0.97 |
| $\gamma$ | Preference parameter | 0.40 |
| $\delta$ | Depreciation rate | 0.06 |
| $\rho$ | TFP autoregresive parameter | 0.95 |
| $\sigma$ | TFP standard deviation | 0.01 |

# The .model file: Define and calibrate parameters

```
!parameters
'Income share of capital' alpha = 0.35
'Discount factor' beta = 0.97
'Preferences parameter' gamma = 0.40
'Depreciation rate' delta = 0.06
'Autorregresive parameter' rho = 0.95
```

- In the `!parameters` section, we declare all parameters (so we can use them later in the equations)
- Optionally, we can calibrate them here, (otherwise we do it in the .m file)

# The .model file: Specify the model equations

```
!transition_equations
'Consumption vs. leisure choice'
C = (gamma/(1-gamma))*(1-L)*(1-alpha)*Y/L;
'Euler equation'
1 = beta * ((C/C{+1}) * (R{+1} + (1-delta)));
'Production function'
Y = A*(K{-1}^alpha) * (L^(1-alpha));
'Capital accumulation'
K = I + (1 - delta) * K{-1};
'Investment equals savings'
I = Y - C;
'Labor demand'
W = (1-alpha) * A * (K{-1} / L)^alpha;
'Capital demand'
R = alpha * A * (L / K{-1})^(1-alpha);
'Productivity AR(1) process'
log(A) = rho * log(A{-1}) + e;
```

- Equations are separated by semicolon
- Lags are indicated by `{-n}`, leads by `{+n}`
- Equations are easier to identify with `'labels'`

# The .m file: Working with the model

- The .m file is a Matlab file, where we work with the model
- To work with IRIS, we need to add it to the path using `addpath`
- It is recommended to start with a clean session
- We read the model using `model`

```
clear all
close all
clc
addpath C:\IRIS
irisstartup()

%% READ MODEL FILE
m = model('torres-
    chapter2.model');
```

# The .m file: Finding the steady state

- IRIS uses the `sstate` command to look for the steady state
- To use it, we have to guess initial values, which we `assign` to the model, starting with the itital params in `get(m,'params')`

```
%% INITIAL VALUES
P = get(m,'params');
P.Y = 1;
P.C = 0.8;
P.L = 0.3;
P.K = 3.5;
P.I = 0.2;
P.W = (1-P.alpha)*P.Y/P.L;
P.R = P.alpha * P.Y/P.K;
P.A = 1;

%% STEADY STATE
m = assign(m, P);
m = sstate(m, 'blocks=',true);
chksstate(m)
get(m,'sstate')
```

# Steady states: results

We find that the steady state is given by

| Variable | Definition | Value | Ratio to $\bar{Y}$ |
|---|---|---|---|
| $\bar{Y}$ | Output | 0.7447 | 1.000 |
| $\bar{C}$ | Consumption | 0.5727 | 0.769 |
| $\bar{I}$ | Investment | 0.1720 | 0.231 |
| $\bar{K}$ | Capital | 2.8665 | 3.849 |
| $\bar{L}$ | Labor | 0.3604 | - |
| $\bar{R}$ | Capital rental price | 0.0909 | - |
| $\bar{W}$ | Real Wage | 1.3431 | - |
| $\bar{A}$ | TFP | 1.0000 | - |

# The .m file: Solving and simulating the model

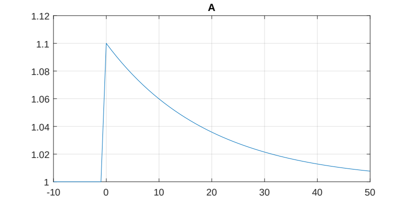
- IRIS uses the `solve` and `simulate` commands to get the solution and run simulations of the model.
- Here, we simulate the impact of an unanticipated 10% increase in total factor productivity:

$$\ln A_t = 0.95 \ln A_{t-1} + \epsilon_t$$

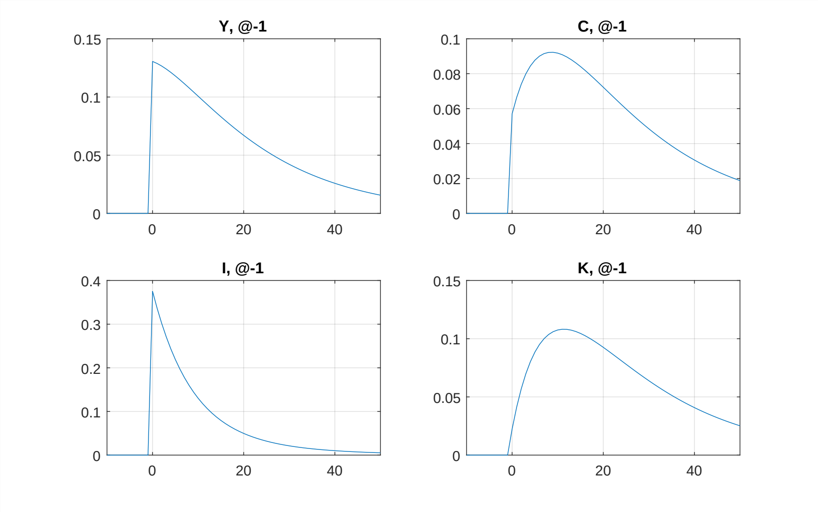
- Notice how persistent the shock is.

```
%% SOLUTION
m = solve(m);

%% SIMULATE PRODUCTIVITY SHOCK
tt = -10:50; %time range
tshock=0; % shock date
d = sstatedb(m, tt);
d.e(0) = 0.10;
s = simulate(m, d, tt,'
    Anticipate=',false);
```

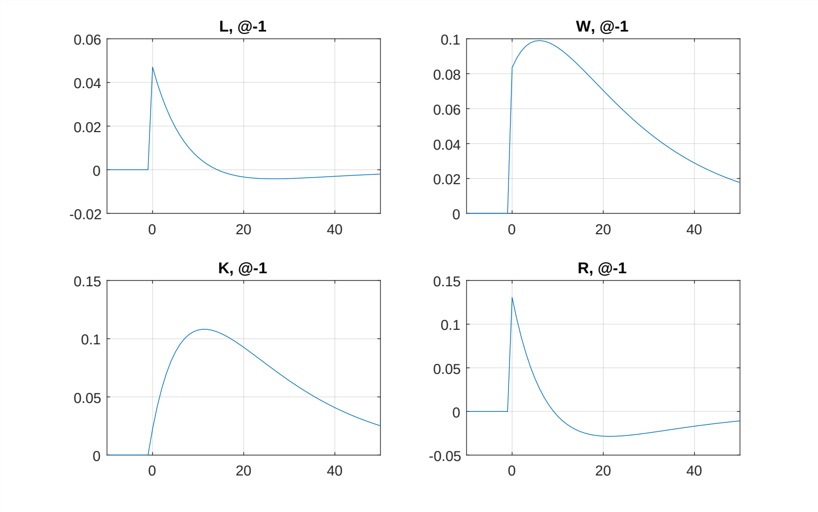


# Responses of endogenous variables to productivity shock




Relative deviations respect to pre-shock values

# Responses of endogenous variables to productivity shock




Relative deviations respect to pre-shock values

## References

Torres, Jose L. (2015). *Introduction to Dynamic Macroeconomic General Equilibrium Models*. 2nd ed. Vernon Press. ISBN: 1622730240.