

Universidad de Costa Rica
EC3201 - Teoría Macroeconómica 2

Practice 3: The Cobb-Douglas utility function

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A consumer problem: 2 goods

A consumer gets utility from consuming goods x and y according to a utility function:

$$U(x, y) \tag{1}$$

The prices of the goods are p_x and p_y , and the consumer has available (nominal) income M . The budget constraint is therefore

$$p_x x + p_y y = M \tag{2}$$

The consumer wants to get as much utility as possible, given the market prices and his income. The Lagrangean for this optimization problem is

$$\mathcal{L}(x, y, \lambda) = U(x, y) + \lambda(M - p_x x - p_y y) \tag{3}$$

where λ is the Lagrange multiplier associated with the budget constraint. In the optimal allocation (x^*, y^*) , it must be the case that

$$U_x(x^*, y^*) = \lambda p_x \tag{4a}$$

$$U_y(x^*, y^*) = \lambda p_y \tag{4b}$$

$$p_x x^* + p_y y^* = M \tag{4c}$$

where U_x and U_y denote the partial derivatives of the utility function with respect to x and to y , respectively.

A Cobb-Douglas utility function

In what follows, we assume that the utility function takes the form of a Cobb-Douglas function:

$$U(x, y) = x^\theta y^{1-\theta} \tag{5}$$

and we define the price index P by

$$P(p_x, p_y) \equiv \left(\frac{p_x}{\theta}\right)^\theta \left(\frac{p_y}{1-\theta}\right)^{1-\theta} \tag{6}$$

where $0 < \theta < 1$.

1. (a) Prove that both the utility function and the price index are homogeneous of degree one.
- (b) Show that the marginal utilities with respect to the goods are given by

$$U_x \equiv \frac{\partial U}{\partial x} = \frac{\theta U}{x} \quad \text{and} \quad U_y \equiv \frac{\partial U}{\partial y} = \frac{(1-\theta)U}{y}$$

[Hint: take logs in both sides of (5)]

- (c) Dividing (4a) by (4b), we find that in the optimal allocation it must be the case that

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

Use this result to show that the optimal allocation x^* and y^* must satisfy the following condition:

$$\frac{x^*}{y^*} = \frac{\theta p_y}{(1-\theta)p_x} \tag{7}$$

- (d) Use (2) and (7) to prove that the demand for goods are given by:

$$x^*(p_x, p_y, M) = \frac{\theta M}{p_x} \quad y^*(p_x, p_y, M) = \frac{(1-\theta)M}{p_y} \tag{8}$$

Explain what this results implies.

- (e) Show that the Lagrange multiplier associated with the budget constraint equals the inverse of the price index, that is: $\lambda^* = P^{-1}$. To do so, you can follow these steps:
 1. Knowing that U is homogeneous of degree one, find an expression for $\frac{U}{y}$. The result should depend on the ratio $\frac{x}{y}$
 2. Use the result of step 1 to compute the marginal utility of y (see question (b)).
 3. Use step 2 and equation (4b) to obtain an expression for λ that depends on $\frac{x}{y}$.
 4. Finally, replace $\frac{x}{y}$ from step 3 with equation (7).
- (f) Use the previous result to show that the partial derivative of the Lagrangean with respect to the nominal income M equals P^{-1} .
- (g) Substitute (8) into (5) to prove that the *indirect utility function*¹ is given by:

$$V(p_x, p_y, M) \equiv U(x^*, y^*) = \frac{M}{P} \tag{9}$$

- (h) Compute the derivative of the indirect utility function with respect to M . Compare your result to that of question f.
- (i) Using (7), show that elasticity of substitution of the goods is given by 1. That is, prove that

$$\frac{\Delta\% \left(\frac{x^*}{y^*} \right)}{\Delta\% \left(\frac{p_x}{p_y} \right)} = -1$$

¹The indirect utility function is a particular case of a *value function*.