

Universidad de Costa Rica
EC3201 - Teoría Macroeconómica 2

Practice 2: The CES utility function

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A consumer problem: 2 goods

A consumer gets utility from consuming goods x and y according to a utility function:

$$U(x, y) \tag{1}$$

The prices of the goods are p_x and p_y , and the consumer has available (nominal) income M . The budget constraint is therefore

$$p_x x + p_y y = M \tag{2}$$

The consumer wants to get as much utility as possible, given the market prices and his income. The Lagrangean for this optimization problem is

$$\mathcal{L}(x, y, \lambda) = U(x, y) + \lambda(M - p_x x - p_y y) \tag{3}$$

where λ is the Lagrange multiplier associated with the budget constraint. In the optimal allocation (x^*, y^*) , it must be the case that

$$U_x(x^*, y^*) = \lambda p_x \tag{4a}$$

$$U_y(x^*, y^*) = \lambda p_y \tag{4b}$$

$$p_x x^* + p_y y^* = M \tag{4c}$$

where U_x and U_y denote the partial derivatives of the utility function with respect to x and to y , respectively.

A CES utility function

In what follows, we assume that the utility function takes the form of a CES function:

$$U(x, y) = (\theta x^\rho + (1 - \theta)y^\rho)^{1/\rho} \tag{5}$$

and we define the price index P by

$$P(p_x, p_y) \equiv [\theta^\sigma p_x^{1-\sigma} + (1 - \theta)^\sigma p_y^{1-\sigma}]^{\frac{1}{1-\sigma}} \tag{6}$$

where $\rho < 1$ and $\sigma \equiv \frac{1}{1-\rho}$.

1. (a) Prove that both the utility function and the price index are homogeneous of degree one.
- (b) Show that the marginal utilities with respect to the goods are given by

$$U_x \equiv \frac{\partial U}{\partial x} = \theta \left(\frac{U}{x} \right)^{1-\rho} \quad \text{and} \quad U_y \equiv \frac{\partial U}{\partial y} = (1-\theta) \left(\frac{U}{y} \right)^{1-\rho}$$

[Hint: raise both sides of (5) to the power of ρ]

- (c) Dividing (4a) by (4b), we find that in the optimal allocation it must be the case that

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

Use this result to show that the optimal allocation x^* and y^* must satisfy the following condition:

$$\frac{x^*}{y^*} = \left(\frac{\theta p_y}{(1-\theta)p_x} \right)^\sigma \tag{7}$$

- (d) Use (2) and (7) to prove that the demand for goods are given by:

$$x^*(p_x, p_y, M) = \left(\frac{\theta}{p_x/P} \right)^\sigma \frac{M}{P} \quad y^*(p_x, p_y, M) = \left(\frac{1-\theta}{p_y/P} \right)^\sigma \frac{M}{P} \tag{8}$$

- (e) Show that the Lagrange multiplier associated with the budget constraint equals the inverse of the price index, that is: $\lambda^* = P^{-1}$. To do so, you can follow these steps:

1. Knowing that U is homogeneous of degree one, find an expression for $\frac{U}{y}$. The result should depend on the ratio $\frac{x}{y}$.
2. Use the result of step 1 to compute the marginal utility of y (see question (b)).
3. Use step 2 and equation (4b) to obtain an expression for λ that depends on $\frac{x}{y}$.
4. Replace $\frac{x}{y}$ from step 3 with equation (7).
5. Finally, simplify the result, keeping in mind that $\sigma \equiv \frac{1}{1-\rho}$.

- (f) Use the previous result to show that the partial derivative of the Lagrangean with respect to the nominal income M equals P^{-1} .

- (g) Substitute (8) into (5) to prove that the *indirect utility function*¹ is given by:

$$V(p_x, p_y, M) \equiv U(x^*, y^*) = \frac{M}{P} \tag{9}$$

- (h) Compute the derivative of the indirect utility function with respect to M . Compare your result to that of question f.

- (i) Using (7), show that elasticity of substitution of the goods is given by σ . That is, prove that

$$\frac{\Delta\% \left(\frac{x^*}{y^*} \right)}{\Delta\% \left(\frac{p_x}{p_y} \right)} = -\sigma$$

- (j) Optional: show that

$$\lim_{\rho \rightarrow 0} U(x, y) = x^\theta y^{1-\theta}$$

that is, that the Cobb-Douglas is a special case of the CES function where $\rho = 0$. Hint: Remember that $\lim_{z \rightarrow 0} g(z) = e^{\lim_{z \rightarrow 0} \ln g(z)}$. Hint 2: You will need L'Hôpital rule.

¹The indirect utility function is a particular case of a *value function*.